MODAL ESTIMATION THEORY FOR DISTRIBUTED PARAMETER SYSTEMS

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MODAL ESTIMATION THEORY FOR DISTRIBUTED

PARAMETER SYSTEMS

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PREFACE

This study is concerned with solving the stochastic estimation problem for a class of linear second order random distributed parameter systems. As the underlying mathematical space is the triple product measure space of time, space, and uncertainty, an approximation scheme is indispensable for a numerically feasible estimation algorithm. The modal estimation theory developed in this study fulfills these needs and clarifies the role of stochastic harmonic analysis in the theory of random distributed parameter systems.

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LIST OF SYMBOLS

$A(\cdot)$	Plant matrix
$B(\cdot)$	Operator which defines the boundary conditions
$C(\cdot)$	Measurement matrix
C_i	Measurement matrix at stage i
$D(\cdot)$	Input matrix for random systems
$D_{i+1,i}$	Input matrix for discretized random model
$E\{\cdot\}$	Expectation operator
$F(i)$	An unknown estimator matrix
$G(\cdot,\cdot)$	Green's function
H	A vector of boundary conditions
$H(i)$	An unknown measurement matrix at stage i
$\hat{J}$	Mean square error criterion
$K(\cdot)$	Filtering gain matrix
$K_s(\cdot)$	Smoothing gain matrix
L	Homogeneous integral equation operator
L^2	Space of functions of square integrable
$M(\cdot)$	Conditional covariance matrix
$P(\cdot)$	Filtering error variance
$P(\cdot\|\cdot)$	Smoothing error variance
$\bar{P}_0$	Initial variance of x(0)
$\bar{P}_b(\cdot)$	Covariance kernel of a periodic process
$Q(\cdot)$	Covariance matrix of a plant noise

$Q_b(\cdot,\cdot)$	Covariance kernel of boundary noise
Q_i	Covariance matrix of a discrete plant noise
$R(\cdot)$	Covariance matrix of a measurement noise
R_i	Covariance matrix of a discrete measurement noise
T	Final time
U	Rate of transport of a physical quantity
$Y(i)$	An unknown measurement vector at stage i
$Z(i)$	A set of measurement up to stage i
a_j	A set of modal coefficients
b_i	i-th sensor constant
c	A wave velocity
d	Sensor spacing interval
$f(\cdot)$	An arbitary function in L^2
$g(\cdot)$	An arbitary function in L^2
h	Depth of an extended boundary
k	Diffusion constant
m	The number of discrete sensor locations
n	The number of modes
$p(\cdot,\cdot)$	Covariance kernel of $x(\cdot)$
p_j	A set of eigenvalues of a homogeneous integral equation
$\bar{p}_0$	A coefficient of initial covariance kernel
$\bar{p}_0(\cdot,\cdot)$	An initial covariance kernel of $u(0,s)$
$p^*(\cdot)$	Finite sum of modal variances
$q(\cdot)$	A scalar covariance function of a plant noise
$r(\cdot)$	A scalar covariance function of a measurement noise
s	Spatial independent variable
t	Temporal independent variable

$u(t,s)$	Solution of a partial differential equation
$u_0(\cdot)$	Initial value of $u(t,s)$
$u_1(\cdot)$	Initial velocity of $u(t,s)$
$\hat{u}(t,s)$	Filtered estimate of $u(t,s)$
$u_s(t,s)$	Spatial derivative of $u(t,s)$
$v(\cdot)$	Measurement noise process
v_i	Discrete measurement noise process
$v_0(\cdot)$	Boundary function at $s=0$
$v_1(\cdot)$	Boundary function at $s=1$
$w(t,s)$	A scalar forcing function
$\bar{w}(\cdot)$	A vector function of modal coefficients $w_j(\cdot)$
$\bar{w}_i$	A vector function of discrete plant noise
$x(\cdot)$	A vector of modal coefficients $x_j(\cdot)$
$x_j(\cdot)$	j-th modal coefficient of $u(t,s)$
$x^*(\cdot)$	Finite mode approximation of $x(\cdot)$
$\hat{x}(\cdot)$	Filtered estimate of $x(\cdot)$
$x^j(\cdot)$	j-th vector of modal coefficients x_{2j-1} and x_{2j}
$\hat{x}(\cdot\|\cdot)$	Smoothed estimate of $x(\cdot)$
$z(\cdot)$	Measurement vector
$\bar{z}(i)$	A set of measurement at stage i
$\hat{z}(T,i)$	The output at the i-th preprocessor
$\Phi(\cdot)$	Transition matrix
$\Phi_{i+1,i}$	Discrete transition matrix
Ω	Sample space
α_j	A monotone decreasing sequence
$\gamma(t,s)$	Stress of an elastic body
$\delta(\cdot)$	An impulse function

$\phi(\cdot)$	A vector of basis functions
$\phi_j(\cdot)$	A set of basis functions
$\rho(s)$	Density of an elastic body
$\lambda(s)$	Modulus of elasticity
λ_j	A set of eigenvalues
ω_j	Angular velocity of j-th mode
$\mu(\cdot,\cdot)$	Expected value of u(t,s)
$\mu(\cdot)$	Expected value of $x(\cdot)$
$\mu_0(\cdot)$	Expected value of u(0,s)
$\mu_1(\cdot)$	Expected value of initial velocity
δ_{ij}	Kronecker delta
$\langle\cdot,\cdot\rangle$	Inner product
$\|\cdot\|$	L^2 norm

CHAPTER I

INTRODUCTION

1.1 Background

Distributed parameter systems arise in many physical or chemical application areas, such as the propagation of electromagnetic fields and sonar waves, the doubly spread targets and channels in communication systems, the wing vibrations and structural beam problems in aerospace systems, the seismic wave propagation in geophysical exploration, and heat conduction, diffusion and transport phenomena in chemical processes. In view of the nature of the physical world evolving in time and space, these are but a few examples of situations wherein the physical variables are defined as a distributed parameter system.

The presence of the additional independent variables in such systems critically affects the modeling and performance of the physical process of concern. For example, in addition to the system having to satisfy some initial conditions, the admissible behavior of the solutions generated by the partial differential equations characterizing the system may be required to satisfy some constraints at the boundaries of the spatial domain of definition. Therefore, there are many subtle mathematical problems that arise when one models distributed parameter systems.

In recent years there has been a growing interest regarding the control and estimation of distributed parameter systems [1]-[19]. This is primarily due to the abundance of potential applications and the rapid advances of mathematical systems theory.

Although much work on the control and estimation of distributed parameter system has been done in order to predict process behavior and to derive a control strategy, the implementation of such algorithms is difficult. Because one is always faced with the necessity of approximating the solution to a set of partial or integro-partial differential equations. Thus, in developing algorithms for distributed parameter systems, the inherent difficulty is at what stage should the approximation be done so that one is able to implement the algorithms easily, and still obtain reasonable results.

Athans [2] pointed out that the distributed nature of the plant should be retained as long as possible to avoid a "property gap" which may exist under the lumped approximation. However, retaining the distributed nature usually yields expressions in the form of sets of vector integro-partial or Riccati-like nonlinear partial differential equations. This creates more difficulty in finding the appropriate approximation schemes and gives the same chances of creating a "property gap", since the information about the behavior of the original physical process, gathered during the process of modeling, is no longer true for the resultant algorithms. The above is especially significant when the stochastic behavior of a random distributed parameter system (D.P.S.) is of interest and the transformation or the adaptation of the a priori statistics to a complex final algorithm is of concern. By approximating at the beginning stage of analysis the statistical properties of the physical process can be incorporated into the approximation procedure. Also early approximation makes it possible to utilize the wealth of knowledge available in the area of lumped parameter systems, such as optimization techniques, state estimation algorithms, or numerical analysis.

In their extensive survey for D.P.S., Polis and Goodson [3] state that "Approximating at the beginning may yield algorithms which are less complex than those derived by retaining the distributed nature of the process." There are two ways of approximating deterministic D.P.S.; one is the finite difference method and the other is the finite mode approximation method. The advantages and disadvantages of each technique are discussed by Polis and Goodson. Although the finite mode approximation technique has many advantages over the finite difference method, the main disadvantages are that the choice of basis functions is not unique, and a poor choice yields a poor approximation. Furthermore, increasing the number of modes not ensure a better approximation if the basis functions are not well chosen. However, for a class of random D.P.S. in this work, this problem can be easily resolved by developing the optimum set of basis functions in the sense of minimum mean square error (MMSE).

In the study of random dynamical systems, the behavior of the system is usually characterized by the propagation of its mean behavior and of its second-moment which describes the amount of uncertainty associated with the mean process. For many applications, to control or to display the behavior of the random system accurately, one needs more detailed information about the physical process of interest. When mere probabilistic description of the process is not adequate, one takes measurements of the process which may contain noise and errors. Estimation is the very process of extracting information from these noisy measurements, taking account of the effects of plant disturbances and prior knowledge of the information. However, the state estimation theory for

D.P.S. is not well-established in spite of many studies that have appeared in literature in the past ten years [4]-[13]. The above is partially due to the fact that the very diverse nature of D.P.S. allows one to approach the problem in many different ways. The work differs both in the formulation of message and measurement process, and in terms of technical approaches such as maximum likelihood [9], least squares [6], or minimum error variance [7]. As far as the computational problem is concerned, most of the work does not consider this important matter.

Sakawa [5] suggested final stage eigenfunction expansion for the Sturm-Liouville equation with the deterministic homogeneous boundary condition as an approximation scheme. But even for the most simple case of homogeneous boundary condition the original consideration of noise at the boundary has to be given up. According to the definition given by Gelb [33], an optimal estimator is a computational algorithm that processes measurements to deduce a minimum error estimate of the state of a system by utilizing: knowledge of the system and measurement dynamics, assumed statistics of system noises and measurement errors, and initial condition information. In view of this, it is unfortunate that most of the estimators for D.P.S. are non-computable, and there is a need to develop computable estimation algorithms for D.P.S..

In this work, the purpose is to represent the solution to certain random partial differential equations in such a way that estimates of the solution can be derived which are computationally feasible. The contribution is really twofold in the sense that new and optimum representations for stochastic D.P.S. are derived, and estimates of the solution of modal representations are developed.

1.2 Objective and Organization

The intent of this study is to find the optimal modal representation for a class of random D.P.S. such as random wave or random diffusion processes and to furnish computationally feasible optimum state estimation algorithms. Utilizing the second moment characterization of the random initial conditions given by the system model, one decomposes the random D.P.S. into a system of stochastic ordinary differential equations. When the autocorrelation function implies a periodic process, Wiener process, or rational spectra, it is interesting that a certain set of orthonormal functions can provide an optimum set of basis functions in the sense of MMSE.

The estimation problem is posed in terms of multiple sensors making measurements on multiple temporal processes at various spatial locations. Applications of state estimation techniques existing for lumped parameter systems to the decomposed modal system are presented under the several assumptions that each system model holds.

In Chapter II, the background material is introduced which is relevant to the development of modal estimation theory in the following chapters. Distributed parameter systems, Karhunen-Loeve expansion of random processes, Kalman filtering, and fixed-interval smoothing algorithms are discussed as essential prerequisites to understanding the material presented in this work.

Chapter III consists of the development of stochastic modal representations for the temporal message model of the random diffusion processes and the temporal or spatial message model for random wave processes. Standard minimization schemes using the properties of L^2

space are employed. The error bounds for each problem are attained.

The estimation problem is formulated and solved in Chapter IV, where optimal techniques of estimation for the temporal and spatial message models with continuous-time and discrete-space observations are developed. When temporal dynamical systems are considered, the linear combination of ordinary Kalman filtered state estimates generates the conditional mean estimate of the solution of random partial differential equations. A correlator type processor, or a Kalman filter estimating a constant, forms the basis for the optimum algorithm for the case of a spatial message model. The overall error bounds of an estimation procedure are investigated to relate the performance measure to the choice of the number of modes.

Chapter V treats an application of the techniques developed in previous chapters to a simple seismic data processing problem. Computer simulation of an example is also included.

A summary and conclusions of the results obtained in the dissertation are presented in Chapter VI where suggestions for further study are also given.

CHAPTER II

PRELIMINARIES

2.1 Introduction

In this chapter, background material from the areas of distributed parameter systems, stochastic Fourier analysis, and state estimation theory is presented. It is intended to provide a self-contained reference for the following chapters in which modal representation of stochastic distributed parameter systems and associated signal processing techniques are developed.

Section 2.2 is concerned with the partial differential equation involving the dynamics of the wave and heat processes. In Section 2.3 the series representation of random processes will be discussed to motivate the parallel development for random fields. Finally, to furnish a necessary background for Chapter IV the state estimation schemes developed for lumped parameter systems will be reviewed.

2.2 Distributed Parameter Systems

Distributed Parameter Systems (D.P.S.) are those systems whose dynamic performance depends not only upon time, but may vary with respect to some other set of independent variables which usually characterize a spatial location in some coordinates. When the dependence of the physical process in question is continuous with respect to the independent variables, it is possible to represent the system by a set of partial differential equations. The general form of a linear second order partial differential equation is:

$$a\frac{\partial^2 u}{\partial t^2} + b\frac{\partial^2 u}{\partial t \partial s} + c\frac{\partial^2 u}{\partial s^2} + d\frac{\partial u}{\partial s} + e\frac{\partial u}{\partial t} + fu = w(t,s) \tag{2.1}$$

where the solution of the equation will be a function u both of time and space, u(t,s). The coefficients are real-valued and twice continuously differentiable functions of t and s. Equation (2.1) can be classified according to its coefficients when a, b, and c do not vanish simultaneously:

1. The equation is of parabolic type at a point (t,s) if $b^2(t,s) = 4a(t,s)c(t,s)$, for example, the inhomogeneous heat or diffusion equation

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial s^2} + w(t,s)$$

2. The equation is of hyperbolic type at a point (t,s) if $b^2(t,s) > 4a(t,s)c(t,s)$, such as the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial s^2}$$

3. The equation is of elliptic type at a point (t,s) if $b^2(t,s) < 4a(t,s)c(t,s)$. An example of this group is Laplce's equation which does not describe a dynamical system, and shall not be given any consideration in this thesis.

If the coefficients a, b, and c are constants, then the equation is of one type in the entire t-s plane. Giving consideration to boundary and initial conditions the heat equation and wave equation are to be discussed in more detail.

2.2.1 The Diffusion or Heat Equation

Let u be some scalar physical quantity defined at every point of a connected interval, $0 \leq s \leq 1$, and U be the rate of transport of u through the interval, then a continuity equation will result:

$$\frac{\partial u}{\partial t} + \operatorname{div} U = w \tag{2.2}$$

where w represents a source term. An obvious example is the diffusion of molecules or ions. If the rate of transport is proportional to the gradient of the quantity u and is directed from large u to small u,

$$U = -k \operatorname{grad} u \qquad k > 0 \tag{2.3}$$

This is physically plausible, since, from kinetic theory, temperature is related to the mean velocity of molecules. The process of heat conduction just represents the diffusion of molecules of greater average speed among those of lower speed. Combining (2.2) with (2.3), one gets the heat or diffusion equation

$$\frac{\partial}{\partial t}u(t,s) = k\frac{\partial^2}{\partial s^2}u(t,s) + w(t,s) \tag{2.4}$$

The function w(t,s) is referred to as the source density which denotes the amount of heat per unit length per unit time generated at the point s at time t.

One needs two boundary conditions and one initial condition to solve

(2.4) for a particular physical situation of concern. Depending on the spatial region of interest, there are two types of problem formulation. The initial-value problem is described as follows:

$$\frac{\partial}{\partial t}u(t,s) = k\frac{\partial^2}{\partial s^2}u(t,s) + w(t,s) \tag{2.5}$$

$$u(0,s) = u_0(s) \tag{2.6}$$

where the spatial variable s is defined in $-\infty < s < \infty$ and $t > 0$. The model describes the flow of heat in a long slender homogeneous bar of uniform cross section, the lateral surface of the bar is insulated and the length is enough to neglect the end effect. The Green's function (G, the impulse response of the homogeneous heat equation) is known to be of the form,

$$G(t,s,t',s') = e^{-(s-s')^2/4k(t-t')}/(4k(t-t'))^{1/2} \tag{2.7}$$

for $t > t'$ and $-\infty < s < \infty$ and

$$G(t,s,t',s') = 0 \tag{2.8}$$

for $t \leq t'$ and $-\infty < s < \infty$, with the following properties [20]:

1. G is continuous on $t \geq t'$ except at the point (t',s') where it has an infinite discontinuity:

$$\lim_{\substack{t-t' \\ t>t'}} G(t,s',t',s') = \lim_{\substack{t-t' \\ t>t'}} \frac{1}{4k(t-t')^{1/2}} = +\infty \tag{2.9}$$

2. G is a solution of the homogeneous heat equation for $t > t'$

3. $\quad G(t,s,t',s')\,ds = 1 \qquad t > t'$ (2.10)

4. $\quad G(t,s',t',s) = G(t,s,t',s') \qquad t > t'$ (2.11)

where t' and s' are parameters independent of t and s. By using the Green's function the integral form of the solution for the problem described by (2.5) and (2.6) can be written as

$$u(t,s) = \int_{-\infty}^{\infty} G(t,s,0,s')u_0(s')ds' + \int_0^t dt' \int_{-\infty}^{\infty} G(t,s,t',s')w(t',s')ds' \tag{2.12}$$

The derivation of the Green's function and the verification of the solution are discussed in [20].

If the spatial region of interest is a finite interval, $0 \le s \le 1$, an initial-boundary-value problem is posed, as described below:

$$\frac{\partial}{\partial t}u(t,s) = k\frac{\partial^2}{\partial s^2}u(t,s) + w(t,s) \qquad 0 < s < 1; \quad t > 0 \tag{2.13}$$

$$u(t,s) = u_0(s) \qquad 0 \le s \le 1; \quad t = 0 \tag{2.14}$$

where the boundary conditions are of the form,

$$\begin{aligned} &\text{Dirichlet:} \quad B(u) = \bar{u} = \begin{bmatrix} v_0(t) \\ v_1(t) \end{bmatrix} \\ &\text{Neumann :} \quad B(u) = \frac{\partial \bar{u}}{\partial n} = \begin{bmatrix} v_0(t) \\ v_1(t) \end{bmatrix} \\ &\text{Mixed} \quad : \quad B(u) = \frac{\partial \bar{u}}{\partial n} + a\bar{u} = \begin{bmatrix} v_0(t) \\ v_1(t) \end{bmatrix} \end{aligned} \tag{2.15}$$

on $s = 0$, $s = 1$, and $t \geq 0$. Boundary conditions may be one of the Dirichlet type, Neumann type, or mixed type as described above, where $B(u)$ represents a boundary operator, $\bar{u}$ and v are vectors, and $\partial\bar{u}/\partial n$ denotes the derivative of $\bar{u}$ in the direction of the exterior normal on the boundary. The parameter a is a known positive constant, and the first element of the vectors refers to $s = 0$, while the second refers to $s = 1$. This problem naturally gives rise to an eigenvalue problem for which the method of eigenfunctions would be suitable [20]. It is possible to express the Green's function in terms of a series expansion:

$$G(t,s,t',s') = \sum_{j=1}^{\infty} \phi_j(s)\phi_j(s')e^{-k\lambda_j(t-t')} \tag{2.16}$$

for $t > t'$ and

$$G(t,s,t',s') = 0 \tag{2.17}$$

for $t \leq t'$, where the set of basis function $\{\phi_j(s)\}$ is assumed to be an orthonormal set obtained by solving the Sturm-Liouville's eigenfunction-eigenvalue problem,

$$\frac{d^2}{ds^2}\phi_j(s) + \lambda_j\phi_j(s) = 0 \qquad 0 \leq s \leq 1 \tag{2.18}$$

$$B(\phi_j) = 0 \qquad s = 0; \quad s = 1 \tag{2.19}$$

and λ_j is the corresponding sequence of eigenvalues. In addition to (2.16) and (2.17), Green's function also satisfies

$$\frac{\partial}{\partial t}G(t,s,t',s') = \frac{\partial^2}{\partial s^2}G(t,s,t',s') + \delta(t-t',s-s')$$

$$0 < s, s' < 1; \quad t, t' > 0 \qquad (2.20)$$

and

$$B(G) = 0 \qquad s = 0; \; s = 1 \qquad (2.21)$$

By means of the Green's function, one can express the solution of initial-boundary problem, (2.13) to (2.14), in terms of the given functions u_0, $w(t,s)$, and $B(u)$. The solution is

$$u(t,s) = \int_0^1 G(t,s,o,s')\, u_0(s')\, ds' + \int_0^t dt' \int_0^1 G(t,s,t',s')\, w(t',s')\, ds'$$

$$- k \int_0^t \left(u \frac{\partial G}{\partial s'} - G \frac{\partial u}{\partial s'}\right) \Bigg|_0^1 dt' \qquad (2.22)$$

If the boundary condition is Dirichlet type, then the last term in (2.22) simplifies to

$$- k \int_0^t H \frac{\partial G}{\partial s'} \Bigg|_0^1 dt' \qquad (2.23)$$

where $B(u) = H$ symbolizes the boundary condition. Similarly, if the boundary condition is Neumann or Mixed type, the last term reduces to

$$+ k \int_0^t HG \Bigg|_0^1 dt' \qquad (2.24)$$

It is worthwhile to note that since the time functions in (2.16) are exponentially decreasing and since $(j)^2$ appears in the arguments of $\{\lambda_j\}$ for many cases, the equation could, for most practical purposes, be approximated, after a short initial period, by its leading terms with $j = 1$.

The existence and uniqueness of the solution for the initial-boundary

-value problem, (2.13) to (2.15), are discussed in [20].

2.2.2. The Wave Equation

Just as the diffusion or heat equation is the most fundamental of parabolic linear partial differential equations, so the most fundamental equation of normal hyperbolic type is the wave equation. It arises in mathematical physics as one of the most important differential equations. From a combination of a continuity equation (2.2) with a second equation describing characteristics of many physical processes, one can get an inhomogeneous wave equation of the form

$$\frac{\partial^2}{\partial t^2} u(t,s) - c^2 \frac{\partial^2}{\partial s^2} u(t,s) = w(t,s) \qquad 0 < s < 1;\ t \geq 0 \quad (2.25)$$

where c denotes a real positive constant. Physically, the solution of (2.25), $u(t,s)$ may represent waves of electric or magnetic intensity, waves of acoustic pressure, transverse or longitudinal displacement waves in a solid, or other phenomena evolving in one dimensional space. The terms $w(t,s)$ and c represent a field source and the velocity of a wave respectively.

In many problems arising in mathematical physics the solution $u(t,s)$ is sought which satisfies prescribed initial conditions (or initial-boundary conditions) as well as the given differential equation. Depending on each prescribed auxiliary condition, the types of problem which are of interest in this study and their solution form may be described as follows:

1). The initial-value problem is characterized by the equations

$$\frac{\partial^2}{\partial t^2} u(t,s) = c^2 \frac{\partial^2}{\partial s^2} u(t,s) + w(t,s) \qquad -\infty < s < \infty \;;\; t \geq 0 \qquad (2.26)$$

$$u(0,s) = u_0(s), \qquad \frac{\partial}{\partial t}u(t,s)\Big|_{t=0} = u_1(s) \qquad -\infty < s < \infty \qquad (2.27)$$

where the solution form (D'Alembert's solution) is known to be

$$u(t,s) = \frac{1}{2}\,[u(s + ct) + u_0(s-ct)] + \frac{1}{2c}\int_{s-ct}^{s+ct} u_1(s')\,ds'$$

$$+ \frac{1}{2c}\int_0^t \int_{s-c(t-t')}^{s+c(t-t')} w(t',s')\,dt'ds' \qquad (2.28)$$

The initial-boundary-value problem with boundary data is described below:

$$\frac{\partial^2}{\partial t^2} u(t,s) = c^2 \frac{\partial^2}{\partial s^2} u(t,s) + w(t,s) \qquad 0 < s < 1;\; t \geq 0 \qquad (2.29)$$

$$B(u) = \begin{bmatrix} v_0(t) \\ v_1(t) \end{bmatrix} \qquad s = 0,\; s = 1;\; t \geq 0 \qquad (2.30)$$

$$u(0,s) = u_0(s), \qquad \frac{\partial}{\partial t}u(t,s)\Big|_{t=0} = u_1(s) \qquad 0 \leq s \leq 1 \qquad (2.31)$$

The equivalent integral representation of the problem through the use of Green's function is given by [27]:

$$u(t,s) = \int_0^1 G(t,s,0,s')\,u_1(s')\,ds' + \int_0^1 \frac{\partial}{\partial t} G(t,s,0,s')\,u_0(s')\,ds'$$

$$- c^2 \int_0^t \left[u(t',s') \frac{\partial}{\partial s'} G(t,s,t',s') - G(t,s,t',s') \frac{\partial u(t',s')}{\partial s'} \Big|_0^1 \right] dt'$$

$$+ \int_0^t \int_0^1 G(t,s,t',s')\,w(t',s')\,ds'\,dt'$$

where the bilinear form for the Green's function of the wave equation,

$$G(t,s,t',s') = \sum_{j=1}^{\infty} \phi_j(s)\ \phi_j(s')\ \frac{\sin \omega_j c(t-t')}{\omega_j c} \qquad (2.33)$$

is the formal solution of the equation (2.29) where

$$w(t,s) = \delta(t-t',s-s') \qquad (2.34)$$

and satisfies the imposed boundary conditions. Also

$$G(t',s,t',s') = 0 \qquad 0 \leq s,s' \leq 1 \qquad (2.35)$$

$$G(t,s',t',s) = G(t,s,t',s') \qquad 0 \leq s,s' \leq 1 \qquad (2.36)$$

$$G(t',s,t,s') = -G(t,s,t',s') \qquad (\text{all } t,t') \qquad (2.37)$$

and $\{\phi_j\}$ denote the onthonormal eigenfunctions associated with the equation:

$$\frac{\partial^2}{\partial s^2}\ \phi_j = -\lambda_j \phi_j \qquad (2.38)$$

where $\lambda_j = \omega_j^2$.

The well-posedness of the initial-value problem described by (2.26), and (2.27), and the existence and uniqueness of the formal series solution of the initial-boundary-value problem, (2.29) to (2.31), are discussed by Dennemeyer [20].

2.3 Series Representation of Stochastic Processes

It is well known that a deterministic function f(t) in $L^2(0,T)$, with inner product $\langle f,g \rangle \triangleq \int_0^T f(t)\, g(t)\, dt$ and norm $\|f\| = \langle f,f \rangle$, may be expanded in a Fourier series with respect to a complete orthonormal set, $\{\phi_j(t)\}$, of elements of $L^2(0,T)$, with the result representing f(t) over any specified finite interval, $0 \leq t \leq T$ [21]. Such an expansion provides the best finite series approximation of the function f(t) in the sense of mean square error, or $L^2(0,T)$ norm, but it is not unique, because the series may be chosen so as to converge to any periodic function which agrees with f(t) on $0 \leq t \leq T$ and the coefficients of any such series are not influenced by the behavior of f(t) outside $0 \leq t \leq T$.

For a random process the situation is similar, but there are differences when the correlation between Fourier coefficients is considered. It is possible to find the optimal n-dimensional basis (in $L^2(0,T)$) for representing particular realizations of a random process such that the $L^2(0,T)$ norm of the error averaged over the ensemble of realization Ω, or $L^2(0,T) \times \Omega$ norm of error, is minimized. The n-dimensional representation for a random process x(t) will take the form

$$x(t) = x^*(t) = \sum_{j=1}^{n} a_j \phi_j(t)$$

$$0 \leq t \leq T \qquad (2.39)$$

where it is assumed, for convenience, that the $\{\phi_j(t)\}$ form an orthonormal basis in $L^2(0,T)$, i.e.,

$$\int_0^T \phi_i(t)\, \phi_j(t)\, dt = \delta_{ij} \qquad 0 \leq t \leq T \qquad (2.40)$$

The "equality" in (2. 39) is to mean precisely that for every t, $0 \leq t \leq T$

$$x(t) = \underset{n \to \infty}{\text{l.i.m.}} \sum_{j=1}^{n} a_j \phi_j(t) \qquad 0 \leq t \leq T \tag{2.41}$$

where l.i.m. (limit in the mean) implies

$$\lim_{n \to \infty} E\left\{ \| x(t) - \sum_{j=1}^{n} a_j \phi_j(t) \|^2 \right\} = 0 \tag{2.42}$$

For the moment the exact form of the ϕ_j will not be specified since the optimum choice will follow from the covariance function of each random process. The $L^2(0,T)$ norm of the error

$$x - x^* = x - \sum_{j=1}^{n} a_j \phi_j \tag{2.43}$$

is to be minimized over the ensemble of sample realization. Then one has

$$\begin{aligned} E\left\{ \|x - x^*\|^2 \right\} &= E\left\{ \langle x - \sum_{j=1}^{n} a_j \phi_j \, , \, x - \sum_{j=1}^{n} a_j \phi_j \rangle \right\} \\ &= E\left\{ \langle x \, , \, x \rangle \right\} + E\left\{ \sum_{j=1}^{n} |\langle x \, , \, \phi_j \rangle - a_j|^2 \right\} \\ &\quad - E\left\{ \sum_{j=1}^{n} |\langle x \, , \, \phi_j \rangle|^2 \right\} \end{aligned} \tag{2.44}$$

where certain well known properties of the inner product have been used [22].

The second expression on the right achieves its minimum when the components a_j have the values

$$a_j = \langle x(t), \phi_j(t)\rangle \qquad 0 \le t \le T ; \quad j = 1, 2, \ldots n \tag{2.45}$$

Then the mean-squared norm of error (2.41) becomes

$$E\left\{\|x - x^*\|^2\right\} = E\left\{ \langle x , x\rangle \right\} - E\left\{\sum_{j=1}^{n} a_j^2\right\} \ge 0 \tag{2.46}$$

which ensures convergence in the mean defined by (2.42) for a second-order random process x(t). Interchanging the expectation and integration operation implied by the inner product, and using (2.44), one has

$$E\left\{\|x - x^*\|^2\right\} = \int E\left\{x^2(t)\right\} dt - \sum_{j=1}^{n}\int_0^T\int_0^T E\left\{x(t)x(t')\right\}\phi_j(t)\phi_j(t)\, dt'dt$$

$$= \int_0^T p(t,t)\, dt - \sum_{j=1}^{n}\int_0^T\int_0^T p(t,t')\phi_j(t)\phi_j(t')\, dt'dt \tag{2.47}$$

Since the first term on the right is independent of the $\{\phi_j(t)\}$, the set of n orthonormal functions which maximize the last term

$$\sum_{j=1}^{n}\int_0^T\int_0^T p(t,t')\phi_j(t)\phi_j(t')\, dt'dt = \sum_{j=1}^{n} \langle L\phi_j , \phi_j\rangle \tag{2.48}$$

is to be chosen where L denotes an integral operator with the covariance function as its kernel; i.e.

$$L\ \phi(t) = \int_0^T p(t,t')\phi(t')\ dt' \tag{2.49}$$

The square integrability, symmetry, and nonnegative definiteness of the covariance function [24] give the following properties of the integral equation [22]:

1). The eigenvalues are real and nonnegative, and form a countable, square summable set.

2). The kernel function can be represented uniformly as an expansion in the eigenfunction $\{\phi_j(t)\}$ of the integral operator L:

$$p(t,t') = \sum_{j=1}^{\infty} p_j\phi_j(t)\phi_j(t') \tag{2.50}$$

3). The eigenfunctions can be made orthonormal and one has

$$<L\ \phi_j\ ,\ \phi_i> \ = \ <p_j\phi_j\ ,\ \phi_i> = p_j\delta_{ji} \tag{2.51}$$

For an operator L with these properties the term on the right of (2.45) can be maximized by a term-by-term maximization, which yields

$$\sum_{j=1}^{n} <L\ \phi_j\ ,\ \phi_j> = \sum_{j=1}^{n} p_j \tag{2.52}$$

where $\{\phi_j\}$ is the set of n eigenfunctions of a homogeneous integral equation,

$$\int_0^T p(t,t')\phi_j(t')\ dt' = p_j\phi_j(t) \qquad 0 \le t \le T\ ;$$

$$j = 1, 2, \ldots n \tag{2.53}$$

with the corresponding n largest eigenvalues. Thus, the performance of this optimal basis, from (2.47), is given by

$$E\left\{\| x - x^* \|^2\right\} = \sum_{j=1}^{\infty} p_j < \phi_j , \phi_j > - \sum_{j=1}^{n} p_j = \sum_{n+1}^{\infty} p_j \tag{2.54}$$

where (2.50) and (2.51) are used. It is noted that the mean squared error is given simply by the sum of the remaining eigenvalues of the integral equation (2.53).

Using this optimal basis, the representation

$$x(t) \simeq x^*(t) = \sum_{j=1}^{n} a_j \phi_j(t) \qquad 0 \le t \le T \tag{2.55}$$

is called the Karhunen-Loeve expansion for a random process [23], [24]. The coefficients in the expansion are uncorrelated random variables, since

$$\begin{aligned} E\left\{a_i a_j\right\} &= E\left\{ <x , \phi_i> <x , \phi_j> \right\} \\ &= \int_0^T \int_0^T p(t,t')\phi_i(t)\phi_j(t')\, dt\, dt' \\ &= p_j\, \delta_{ij} \qquad i, j = 1, 2, \ldots n \end{aligned} \tag{2.56}$$

where (2.39) and (2.50) are utilized. If x is a zero-mean process, then the coefficients are also zero mean. If x is not zero mean, one may add a fixed term $\mu(t)$ to the representation, i.e.

$$x^*(t) = \sum_{j=1}^{n} a_j \phi_j(t) + \mu(t) \tag{2.57}$$

where $\mu(t)$ is simply the mean value $E\{x(t)\}$ for the process.

Davenport [25] discusses techniques for solving the integral equation (2.53) with stationary covariance kernels having spectral densities which are rational functions of frequency. State variable solution methods for various kernels and several particular types of non-stationary kernels of interest have been presented in [26] and [23], respectively.

2.4 Linear State Estimation Theory

State estimation techniques for linear dynamical systems are considered to be pertinent to the modal estimation theory studied in Chapter V. Among the areas of interest are Kalman filtering, and linear fixed interval smoothing theory. A brief description of the main aspects of these are reviewed in this section.

2.4.1 Kalman Filtering [33]-[34]

This technique is concerned with estimating the state of a linear dynamical system, given noisy linear measurements of the state. Consider a system described by the formal stochastic differential equation

$$\dot{x}(t) = A(t)x(t) + D(t)\bar{w}(t) \tag{2.58}$$

where $x(t)$ is an n vector representing the state of the system and $\bar{w}(t)$ is a zero mean white noise vector, often referred to as plant noise or process noise. The covariance expression associated with $\bar{w}(t)$ is

$$E\left\{\bar{w}(t)\bar{w}^T(t')\right\} = Q(t)\,\delta(t-t') \tag{2.59}$$

At each instant of time from 0 up to the current time, a measurement process is available

$$z(t) = C(t)x(t) + v(t) \tag{2.60}$$

where z is a m dimensional measurement vector, C is a m x n measurement matrix, and v is a measurement noise vector of zero mean white noise with a covariance matrix

$$E\left\{v(t)v^T(t')\right\} = R(t)\,\delta(t-t') \tag{2.61}$$

The noise processes $\bar{w}$ and v are statistically independent with each other or with the initial state $x(0)$ which is a Gaussian random vector with mean

$$E\left\{x(0)\right\} = \mu_0 \tag{2.62}$$

and variance

$$\mathrm{Var}\left\{x(0)\right\} = P_0 \tag{2.63}$$

Given an observation z, the Kalman filter is used to generate the estimate $\hat{x}(t)$ of the state $x(t)$. When the initial conditions and noise terms are Gaussian, the filter generates the conditional mean estimate

$$\hat{x}(t) = E\left\{x(t) \mid z(t') : \quad 0 \le t' \le t\right\} \tag{2.64}$$

Even if these terms are nongaussian, the filter still provides the best linear estimate in the minimum mean square error sense. The filter estimate is generated by the equation

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + K(t)[z(t) - C(t)\hat{x}(t)] \tag{2.65}$$

where $K(t)$ is referred to as the Kalman gain. The gain is obtained from the expression

$$K(t) = P(t)C^T(t)R^{-1}(t) \tag{2.66}$$

where the error variance matrix $P(t)$ is defined as

$$\dot{P} = A(t)P(t) + P(t)A^T(t) - P(t)C^T(t)R^{-1}(t)C(t)P(t)$$

$$+ D(t)Q(t)D^T(t) \tag{2.67}$$

The initial conditions for (2.65) and (2.67) are

$$\hat{x}(0) = \mu_0 \tag{2.68}$$

and

$$P(0) = P_0 \tag{2.69}$$

For the discrete version of Kalman filter [34], the dynamical system can be put in the form

$$x_{i+1} = \Phi_{i+1,i}x_i + D_{i+1,i}\bar{w}_i \qquad i = 0, 1, \ldots m \tag{2.70}$$

where x_i is an n vector representing the state of the system at stage i and the initial state x_0 is a Gaussian random vector with mean

$$E\left\{x_0\right\} = \mu_0 \tag{2.71}$$

and

$$\mathrm{Var}\left\{x_o\right\} = P_0 \tag{2.72}$$

The plant noise $\bar{w}_i$ is zero mean discrete white noise with covariance description

$$E\left\{\bar{w}_i\bar{w}_k^T\right\} = Q_i\ \delta_{ik} \tag{2.73}$$

Noisy observations of the state vector are available given by the linear observation model

$$z_{i+1} = C_{i+1}x_{i+1} + v_{i+1} \qquad i = 0, 1, \ldots m \tag{2.74}$$

where z_{i+1} is the m dimensional observation vector at stage i+1 and v_{i+1} is zero mean discrete white noise with covariance expression

$$E\left\{v_i v_k^T\right\} = R_i \, \delta_{ik} \tag{2.75}$$

Given a set of measurement data $Z_i = \{z_1, z_2, \text{---} \, z_i\}$ and a new measurement z_{i+1}, the optimal filtered estimate $\hat{x}_{i+1}$ is given by the recursive relation

$$\hat{x}_{i+1} = \Phi_{i+1,i}\hat{x}_i + K_{i+1}[z_{i+1} - C_{i+1}\Phi_{i+1,i}\hat{x}_i] \tag{2.76}$$

for i = 0,1,2, --- , where $\hat{x}_i$ is the conditional mean estimate if the noise and initial conditions are Gaussian, i.e.

$$\hat{x}_i = E\left\{x_i \;\middle|\; Z_i\right\} \tag{2.77}$$

The initial condition for (2.76) is $\hat{x}_0 = \mu_0$. The discrete Kalman gain is obtained from the expression

$$K_{i+1} = M_{i+1}C_{i+1}^T[R_{i+1} + C_{i+1}M_{i+1}C_{i+1}^T]^{-1} \tag{2.78}$$

where the predicted conditional covariance M_{i+1} is given by

$$M_{i+1} \triangleq \Phi_{i+1,i}P_i\Phi_{i+1,i} + D_{i+1,i}Q_i \, D_{i+1,i}^T \tag{2.79}$$

and the updated conditional covariance matrix P_i satisfies the equation

$$P_{i+1} = [I - K_{i+1} C_{i+1}]M_{i+1}[I - K_{i+1} C_{i+1}]^T + K_{i+1}R_{i+1}K_{i+1}^T \quad (2.80)$$

with $P_0 = \mathrm{Var}\{x_0\}$. Under non-Gaussian conditions the filter provides the best linear estimate in the minimum mean square error sense, but does not give the conditional mean estimate.

There is another version of the Kalman filter which is referred to as the continuous-discrete version. This problem formulation refers to the situation where the dynamics are described by a continuous time model, indicated by (2.58), but the observations are discrete in time, as indicated by (2.74). The best estimate is then described in between observations at time t_i and t_{i+1} by the free message model

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) \quad (2.81)$$

with initial condition at time t_i

$$\hat{x}(t_i) = \hat{x}(t_i^+) \quad (2.82)$$

where the "+" indicates the updated estimate at time t_i. The updating is obtained according to the procedure

$$\hat{x}(t_i^+) = \hat{x}(t_i) + K_i[z_i - C_i\hat{x}(t_i)] \quad (2.83)$$

The $\hat{x}(t_i)$ on the right hand side of (2.83) is obtained from integrating (2.81) from the previous condition, $\hat{x}(t_{i-1}^+)$. Thus the filtered estimate is discontinuous at observation instants.

The gain K_i is evaluated according to the formula

$$K_i = M_i C_i^T [R_i + C_i M_i C_i^T]^{-1} \tag{2.84}$$

where $M_i = M(t_i)$ is calculated by integrating the equation

$$M(t) = A(t)M(t) + M(t)A^T(t) + D(t)Q(t)D^T(t) \tag{2.85}$$

from previous condition

$$M(t_{i-1}) = P_{i-1} \tag{2.86}$$

and P_i is calculated as indicated by Eq. (2.80). The initial conditions are expressed by

$$\hat{x}_{i0} = E\left\{x_{i0}\right\} \tag{2.87}$$

$$P_{i0} = \mathrm{Var}\left\{x_{i0}\right\} \tag{2.88}$$

The various filtering algorithms which have been summarized here are primarily used in an "on-line" manner. That is, an estimate of the state of a system at time t_i, given measurements from time t_0 to time t, is obtained. However, if an estimate of the state at some time t, given measurements up to $T > t$, is the question of concern, then this leads one to the algorithms referred to as fixed interval smoothing algorithms. These algorithms make use of the filtering results and generally provide an improved estimate, i.e. better than the filtered estimate.

2.4.2 Fixed Interval Smoothing [49]

In this section the linear fixed interval smoothing algorithm is summarized. The dynamical description is as indicated by Eq. (2.58) and the observational model is as indicated by (2.60). Observations are assumed to have been made during a fixed interval [0,T], i.e. the data set is

$$Z_T = \left\{ z(t) \; : \; t \in [0,T] \right\} \tag{2.89}$$

and estimates are to be made of the state of the system during that time interval. The essential difference between the filtered estimate and the smoothed estimate is that for the smoothed estimate, the estimate of the state at time t is obtained on the basis of the data set Z_T. In the case where the noise is gaussian and where the initial conditions are Gaussian, then the smoothed estimate denoted as $\hat{x}(t|T)$ is the conditional mean estimate

$$\hat{x}(t|T) = E\left\{ x(t) \;\middle|\; Z_T \right\} \tag{2.90}$$

In the non-Gaussian case, the smoothed estimate is the best linear estimate in the minimum mean square error sense. The smoothed estimate for the system described by Eqs. (2.58) and (2.60) is governed by the relation.

$$\dot{\hat{x}}(t|T) = A(t)\hat{x}(t|T) + K_s(t)[\hat{x}(t|T) - \hat{x}(t)] \tag{2.91}$$

for $0 \leq t \leq T$ where $\hat{x}(t)$ is the optimal filtered estimate as generated by (2.65), and $K_s(t)$ is the "smoothing filter" gain matrix. Equation (2.91) is integrated backward in time from the terminal condition

$$\hat{x}(t|T) = \hat{x}(T) \tag{2.92}$$

The smoothing filter gain and error variance are given by the relations

$$K_s(t) = D(t)Q(t)D^T(t)P^{-1}(t) \tag{2.93}$$

and

$$\dot{P}(t|T) = [A(t) + K_s(t)]P(t|T) + P(t|T)[A(t) + K_s(t)]^T - D(t)Q(t)D^T(t) \tag{2.94}$$

where $P(t)$ is as calculated by (2.67). As indicated in (2.91) the filtered estimate is linked to the smoothed estimate as an input-output relationship. Considerable use of the filtering and smoothing algorithms summarized so far will be made in the remaining chapters.

2.5 Summary

In this chapter some mathematical preliminaries, vital to the development of the subsequent chapters, has been collected. The Green's function solution form for the initial-boundary-value problem for heat and wave processes has been presented. The Karhunen-Loeve expansion of random processes has been emphasized as the basic concept to be extended to the series representation of a random field. Linear state estimation algorithms, such as Kalman filtering and fixed-interval smoothing theory were introduced to be a self-contained reference for the study of modal estimation in Chapter IV.

CHAPTER III

MODAL REPRESENTATION

3.1 Introduction

The eigenfunction expansion method for a deterministic D.P. system plays an important role in studying such systems analytically or numerically. In this chapter the method of modal representation is to be extended to stochastic D.P. systems. The purpose of the chapter is to develop those aspects of modal representation which are specific to the decomposition of a random partial differential equation to a set of random ordinary differential equations.

In Section 3.2 a second-order linear stochastic partial differential equation is introduced in its most general form. The random heat equation and random wave equation are presented as a special case of this general form. Section 3.3 is concerned with the topic of modal representation for random fields. Of particular interest is the series expansion of a white gaussian random field. This subject is discussed in some detail since it serves as a part of mathematical justification of the modal representation technique.

In Section 3.4, the representation of temporal modes is studied for random diffusion or heat equations, which are to be decomposed to a set of stochastic ordinary differential equations with random initial conditions. Given a priori statistics of the system, an optimal modal representation is achieved for a class of important systems. Section 3.5 deals with the similar decomposition of the random wave processes. Chapter III concludes with the study of error bounds for modal representation.

3.2 Random Distributed Parameter Systems

Compared to the field of stochastic ordinary differential equations, relatively little is known concerning random partial differential equations. Of course, many of the results from lumped parameter systems are easily generalized to distributed parameter systems. Since it is not difficult to define white noise with a multidimensional parameter [24], there have been many attempts to formulate random distributed parameter problems [4-13] and to study the nature of random fields [14-19]. In this work, stochastic partial differential equations are investigated with the work based in part on the above investigations.

Continuous dynamical systems with distributed parameters (D.P. systems), which are subject to disturbances have been represented by a second-order partial differential equation in Section 2.2. If one lets u(t,s) and w(t,s) be, respectively, the state of a random field and the random disturbances in a sample space Ω at time t and space s, then a general second-order random partial differential equation may be written as

$$a\frac{\partial^2 u}{\partial t^2} + b\frac{\partial^2 u}{\partial t \partial s} + c\frac{\partial^2 u}{\partial s^2} + d\frac{\partial u}{\partial s} + e\frac{\partial}{\partial t} + fu = w(t,s) \qquad (3.1)$$

where the coefficients a,b,c,d,e,f are real valued and twice continuously differentiable functions of t and s. The random forcing function, w(t,s), is a two dimensional aussian random field with the following statistical properties:

$$E\left\{w(t,s)\right\} = 0 \tag{3.2}$$

$$E\left\{w(t,s)w(t',s')\right\} = Q(t,s,t',s') \tag{3.3}$$

where Q(t,s,t',s') represent the covariance function of w(t,s). As is the case of the deterministic problem, (3.1) may also be classified according to its coefficients a,b and c. If the coefficients are random variables, it may happen that the equation is not of one type in the entire t-s plane. There are many ways of formulating random D.P. systems depending upon the source of randomness introduced into the physical process. One of the coefficients influencing the system may be subject to random fluctuations yielding random parameter partial differential equation. Deterministic information may not be available about the forcing term, initial, or boundary conditions, so that a deterministic formulation of the initial or initial-boundary problem is not feasible. Considering such situations, one can have the following types of random problem:

1). Random coefficient problem [3]

2). Random initial-value problem [13,45]

3). Random boundary-value problem [48]

4). Random forcing-function problem [6-12]

5). Mixed-type (Random initial, boundary, and forcing function) problem [5]

Since the random coefficient problem usually yields a nonlinear filtering problem, it is not given further consideration in this study. In the literature of filtering, Sakawa's work [5] is the first attempt to include random boundary conditions. The most general form of random D.P. systems including random initial-boundary conditions and random forcing term is to be studied for the diffusion or heat and wave processes in the sequel. It is noted that the sample space Ω of the random initial-value problem to be discussed in this study is assumed to be the set of all solutions to well-posed problems, in the sense that the solution exists uniquely for each sample of initial and boundary data, and forcing function, and depends continuously on these terms.

3.2.1 Stochastic Diffusion or Heat Equation

The homogeneous random initial-value problem studied by Kampe De Feriet [45] for the statistical theory of turbulence is one of the first cases of stochastic modeling of the diffusion process. Here, the inhomogeneous random equation is formulated as

$$\frac{\partial}{\partial t} u(t,s) = k\frac{\partial^2}{\partial s^2} u(t,s) + w(t,s)$$

$$-\infty < s < \infty \quad ; \quad t > 0 \tag{3.4}$$

where k is a known real valued constant and w(t,s) is a white Gaussian random field with statistics given by (3.2) and (3.3). A random initial condition with Gaussian statistics

$$E\left\{u(0,s)\right\} = \mu_0(s) \qquad -\infty < s < \infty \tag{3.5}$$

$$E\left\{[u(0,s) - \mu_0(s)][u(0,s') - \mu_0(s')]\right\} = \bar{p}_0(s,s') \tag{3.6}$$

is defined where $\mu_0(s)$ is an expected value of the initial state and $\bar{p}_0(s,s')$ is an associated covariance function. It is assumed that $u(0,s)$ and $w(t,s)$ are uncorrelated for all $-\infty<s<\infty$ and $t > 0$. From the solution form of Green's function, (2.10), the first moment is found to be

$$\mu(t,s) = E\{u(t,s)\}$$

$$= E\left\{\int_{-\infty}^{\infty} G(t,s,0,s')u_0(s')\, ds' + \int_0^t dt' \int_{-\infty}^{\infty} G(t,s,t',s')w(t',s')\, ds'\right\} \tag{3.7}$$

By interchanging the integration and expectation operations and knowing that the function $G(t,s,t',s')$ is a deterministic quantity, one obtains

$$\mu(t,s) = \int_{-\infty}^{\infty} G(t,s,0,s')\mu_0(s')\, ds' \tag{3.8}$$

The variance may be obtained similarly,

$$\bar{p}(t_1,s_1,t_2,s_2) \triangleq \mathrm{Cov}\left\{u(t_1,s_1),u(t_2,s_2)\right\}$$

$$= \mathrm{Cov}\left\{\int_{-\infty}^{\infty} G(t_1,s_1,0,s_1')u_0(s_1')\, ds_1'\right.$$

$$+ \int_0^{t_1} dt_1' \int_{-\infty}^{\infty} G(t_1,s_1,t_1',s_1')w(t_1',s_1')\, ds_1' ,$$

$$\int_{-\infty}^{\infty} G(t_2,s_2,0,s_2')u_0(s_2')\, ds_2' + \int_0^{t_2} dt_2' \int_{-\infty}^{\infty} G(t_2,s_2,\ t_2',s_2')w(t_2',s_2')\, ds_2' \Big\} \quad (3.9)$$

Once again interchanging the expectation and integration operations, and using the assumption that $u(0,s)$ and $w(t,s)$ for $\infty < s < \infty$; $t \geq 0$ are uncorrelated, one obtains

$$\bar{p}(t_1,s_1,t_2,s_2) = \int_{-\infty}^{\infty} ds_1' \int_{-\infty}^{\infty} ds_2' G(t_1,s_1,0,s_1')p(s_1',s_2')G(t_2,s_2,0,s_2')$$

$$+ \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \int_{-\infty}^{\infty} ds_1' \int_{-\infty}^{\infty} ds_2' G(t_1,s_1,t_1',s_1')Q(t_1',s_1',t_2',s_2')G(t_2,s_2,t_2',s_2') \quad (3.10)$$

When the region of interest is a closed interval, $0 \leq s \leq 1$, the stochastic version of the inhomogeneous initial-boundary-value problem given by (2.11) - (2.13) may be formulated. In addition to the equations, (3.4), (3.5) and (3.6), with a new definition of spatial region, $0 \leq s \leq 1$, there needs to be an additional condition at the two boundary points, $s = 0$ and $s = 1$. For the simplicity of treatment $\mu_0(s)$ in (3.5) is assumed to be zero. The mixed boundary data problem is considered without loss of generality:

$$\frac{\partial}{\partial s'} u(t,s')\Big|_{s'=0} + a_0 u(t,0) = v_0(t) \qquad t \geq 0 \qquad (3.11)$$

$$\frac{\partial}{\partial s'} u(t,s')\Big|_{s'=1} + a_1 u(t,1) = v_1(t) \qquad t \geq 0 \qquad (3.12)$$

where a_0 and a_1 are nonnegative real constants and $v_0(t)$ and $v_1(t)$ are white Gaussian random processes with the following statistical properties,

$$E\left\{\begin{bmatrix} v_0(t) \\ v_1(t) \end{bmatrix}\right\} = 0 \tag{3.13}$$

$$E\left\{\begin{bmatrix} v_0(t) \\ v_1(t) \end{bmatrix} [v_0(t) \;\; v_1(t)]\right\} = P_b(t,t') \tag{3.14}$$

The white Gaussian processes $w(t,s)$, $v_0(t)$ and $v_1(t)$ are assumed to be statistically independent of each other and also independent of the random initial condition $u(0,s)$. The solution form is given by (2.19), when Green's function $G(t,s,t',s')$ for the stochastic problem is defined for $t,t' \geq 0$ and $0 \leq s,s' \leq 1$, in the same manner as in Section 2.2.1 such that

1. $$\frac{\partial}{\partial t}G(t,s,t',s') - \frac{\partial^2}{\partial s^2}G(t,s,t',s') = \delta(t - t',s - s') \tag{3.15}$$

2. $$G(t,s,t',s') = 0 \qquad t < t' \; ; \;\; 0 \leq s,s' \leq 1 \tag{3.16}$$

3. $$\left.\frac{\partial}{\partial s}G(t,s,t',s')\right|_{s=0} + a_0 G(t,0,t',s') = 0 \tag{3.17}$$

$$\left.\frac{\partial}{\partial s}G(t,s,t',s')\right|_{s=1} + a_1 G(t,1,t',s') = 0 \tag{3.18}$$

Then the first moment may be obtained by taking the expectation of both sides of (2.19) considering w(t,s) and u(t,s) as random quantities:

$$\mu(t,s) = E\{u(t,s)\} = E\left\{ \int_0^1 G(t,s,0,s')u(0,s')\, ds' + \right.$$

$$\left. \int_0^t dt' \int_0^1 G(t,s,t',s')w(t',s')\, ds' - k \int_0^t \left(u\frac{\partial G}{\partial s'} - G\frac{\partial u}{\partial s'} \right)\Bigg|_0^1 dt' \right\} \qquad (3.19)$$

By interchanging the expectation and the integration operation and by using (3.11), (3.12), (3.17), and (3.18) one obtains

$$\mu(t,s) = \int_0^1 G(t,s,0,s')\mu_0(s')\, ds' \qquad (3.20)$$

where (3.2) and (3.5) are also utilized.

The covariance may also be determined by defining

$$\bar{p}(t_1,s_1,t_2,s_2) \triangleq \mathrm{Cov}\{u(t_1,s_1)u(t_2,s_2)\} \qquad (3.21)$$

Substituting (2.19) into (3.21) and making use of (3.3), (3.6), and (3.14), one obtains

$$\bar{p}(t_1,s_1,t_2,s_2) = \int_0^1 ds_1' \int_0^1 ds_2' G(t_1,s_1,0,s_1')\bar{p}(s_1',s_2')G(t_2,s_2,0,s_2')$$

$$+ k^2 \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \Big[G(t_1,s_1,t_1',0)\, G(t_2,s_2,t_2',0) q_{00}(t_1',t_2') + G(t_1,s_1,t_1',1) q_{01}(t_1',t_2')$$

$$+ G(t_1,s_1,t_1',1)\, G(t_1,s_1,t_1',0) q_{10}(t_1',t_2') + G(t_1,s_1,t_1',1) q_{11}(t_1',t_2') \Big]$$

$$+ \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \int_0^{1} ds_1' \int_0^{1} ds_2' G(t_1,s_1,t_1',s_1') Q(t_1',s_1',t_2',s_2') G(t_2,s_2,t_2',s_2') \quad (3.22)$$

where $q_{00}(t_1',t_2')$, $q_{01}(t_1',t_2')$, $q_{10}(t_1',t_2')$ and $q_{11}(t_1',t_2')$ are components of $Q_b(t_1',t_2')$. The computation associated with (3.22) is not feasible unless the exact expression for Green's function is known. As discussed in Section 2.2.1 Green's function may be expressed in terms of orthonormal coordinate functions $\{\phi_j(s)\}$:

$$G(t,s,t',s') = \sum_{j=1}^{\infty} \phi_j(s)\, \phi_j(s') e^{-k\lambda_j (t-t')} \quad (3.23)$$

where λ_j is defined by (2.16). For the stochastic problem, it is possible to find the optimum set of basis function $\{\phi_j(s)\}$. This will be discussed in Section 3.4 using expression (3.23).

3.2.2 Stochastic Wave Equation

The random counterpart of the deterministic wave equations discussed in Section 2.2 may also be formulated by introducing randomness in its inputs and in its initial or boundary conditions. A problem of this type defined in an unbounded spatial region, $-\infty < s < \infty$, is denoted as a random initial-value wave problem,

$$\frac{\partial^2}{\partial t^2}u(t,s) = c^2\frac{\partial^2}{\partial s^2}u(t,s) + w(t,s) \tag{3.24}$$

$$E\{u(0,s)\} = E\{u_0(s)\} = \mu_0(s) \tag{3.25}$$

$$E\{\frac{\partial}{\partial t}u(t,s)\Big|_{t=0}\} = E\{u_1(s)\} = \mu_1(s) \tag{3.26}$$

$$-\infty < s < \infty \ ; \quad t \geq 0$$

where the covariance of the initial conditions is

$$E\left\{\begin{bmatrix} u(0,s) - \mu_0(s) \\ \frac{\partial}{\partial t}u(t,s)\Big|_{t=0} - \mu_1(s)\end{bmatrix}\left[u(0,s') - \mu_0(s') \quad \frac{\partial}{\partial t}u(t,s')\Big|_{t=0} - \mu_1(s')\right]\right\}$$

$$= \bar{P}_0(s,s') \qquad -\infty < s,s' < \infty \tag{3.27}$$

and $w(t,s)$ is characterized by (3.2) and (3.3). From the D'Alembert solution form of (2.27) one obtains the first moment

$$E\{u(t,s)\} = \mu(t,s) = E\left\{\frac{1}{2}[u_0(s + ct) + u_0(s - ct)]\right\}$$

$$+ E\left\{\frac{1}{2c}\int_{s-ct}^{s+ct} u_1(s')ds'\right\} + E\left\{\frac{1}{2c}\int_0^t dt'\int_{s-c(t-t')}^{s+c(t-t')} ds'\ w(t',s')\right\} \tag{3.28}$$

which becomes

$$\mu(t,s) = \frac{1}{2}\,[\mu_0(s+ct) + \mu_0(s-ct)] + \frac{1}{2c}\int_{s-ct}^{s+ct}\mu_1(s')ds' \qquad (3.29)$$

The variance of the solution is given by

$$\bar{p}(t_1,s_1,t_2,s_2) \triangleq \mathrm{Cov}\{u(t_1,s_1)\ ,\ u(t_2,s_2)\}$$

$$= E\Big\{[u(t_1,s_1) - \mu(t_1,s_1)][u(t_2,s_2) - \mu(t_2,s_2)]\Big\} \qquad (3.30)$$

From (2.27) and (3.22) it follows that

$$\bar{p}_{11}(t_1,s_1,t_2,s_2) = \frac{1}{4}\,\bar{p}_0^{11}(s_1+ct_1,s_2+ct_2) + \frac{1}{4}\,\bar{p}_0^{11}(s_1-ct_1,s_2+ct_2)$$

$$+ \frac{1}{2}\,\bar{p}_0^{11}(s_1+ct_1,s_2+ct_2) + \frac{1}{4c^2}\int_{s_1-ct_1}^{s_1+ct_1} ds_1' \int_{s_2-ct_2}^{s_2+ct_2} ds_2' p_0^{22}(s_1',s_2')$$

$$+ \frac{1}{4c^2}\int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \int_{s_1-c(t_1-t_1')}^{s_1+c(t_1-t_1')} ds_1' \int_{s_2-c(t_2-t_2')}^{s_2+c(t_2-t_2')} ds_2'\ Q(t_1',s_1',t_2',s_2')$$

$$+ E\Big\{\frac{1}{4c}[u_0(s_1+ct_1) - \mu_0(s_1+ct_1) + u_0(s_1-ct_1)-\mu_0(s_1-ct_1)]$$

$$\cdot\int_{s_2-ct_2}^{s_2+ct_2} [u_1(s_2')-\mu_1(s_2')]ds_2' + \frac{1}{4c}\int_{s_1-ct_1}^{s_1+ct_1} [u_1(s_1')-\mu_1(s_1')]\ ds_1'$$

$$\cdot [u_0(s_2+ct_2) - \mu_0(s_2+ct_2) + u_0(s_2-ct_2) - \mu_0(s_2-ct_2)] \Big\} \quad (3.31)$$

Rearranging the last two terms, one obtains

$$\bar{p}_{11}(t_1,s_1,t_2,s_2) = \frac{1}{4}\Bigg[\bar{p}_0^{11}(s_1+ct_1,s_2+ct_2) + \bar{p}_0^{11}(s_1-ct_1,s_2+ct_2)$$

$$+ 2p_0^{11}(s_1+ct_1,s_2-ct_2) + \frac{1}{c^2}\int_{s_1-ct_1}^{s_1+ct_1} ds_1' \int_{s_2-ct_2}^{s_2+ct_2} ds_2'\, p_0^{22}(s_1',s_2')$$

$$+ \frac{1}{c}\int_{s_2-ct_2}^{s_2+ct_2} [\bar{p}_0^{12}(s_1+ct_1,s_2') + \bar{p}_0^{12}(s_1-ct_1,s_2')]\, ds_2'$$

$$+ \frac{1}{c}\int_{s_1-ct_1}^{s_1+ct_1} [\bar{p}_0^{21}(s_2+ct_2,s_1') + \bar{p}_0^{21}(s_2-ct_2,s_1')]\, ds_1'$$

$$+ \frac{1}{c^2}\int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \int_{s_1-c(t_1-t_1')}^{s_1+c(t_1-t_1')} ds_1' \int_{s_2-c(t_2-t_2')}^{s_2+c(t_2-t_2')} ds_2' Q(t_1',s_1',t_2',s_2')\Bigg] \quad (3.32)$$

where $\bar{p}_0^{11}$, $\bar{p}_0^{22}$, $\bar{p}_0^{12}$, and $\bar{p}_0^{21}$ are elements of the matrix $\bar{P}_0$.

The random initial-boundary-value wave problem associated with (3.24) through (3.27) may also be formulated with the change of the spatial region of interest, from $-\infty < s < \infty$ to $0 \leq s \leq 1$, and with the addition of boundary conditions defined by (3.11) through (3.14). The equivalent statistical properties are also assumed to hold. The solution form may also be obtained from (2.31) where $u_0(s)$, $u(t,0)$, $u(t,1)$, and $w(t,s)$ are random quantities and the Green's function is defined to satisfy

$$1. \quad \frac{\partial^2}{\partial t^2} G(t,s,t_1,s_1) - \frac{\partial^2}{\partial s^2} G(t,s,t_1,s_1) = \delta(t-t_1, s-s_1) \tag{3.33}$$

when $0 \leq s,s' \leq 1$ and $t,t' \geq 0$

$$2. \quad G(t,s,t_1,s_1) = 0 \qquad 0 \leq s,s_1 \leq 1 \ ; \quad t < t_1 \tag{3.34}$$

$$3. \quad \frac{\partial}{\partial s} G(t,s,t_1,s_1)\bigg|_{s=0} + a_0 G(t,0,t_1,s_1) = 0 \tag{3.35}$$

$$\frac{\partial}{\partial s} G(t,s,t_1,s_1)\bigg|_{s=1} + a_1 G(t,1,t_1,s_1) = 0 \tag{3.36}$$

Taking the expected value of both sides of (2.31), one obtains the mean value of $u(t,s)$,

$$\mu(t,s) \triangleq E\{u(t,s)\}$$

$$= E\left\{\int_0^1 G(t,s,0,s')u_1(s')\,ds' + \int_0^1 \frac{\partial}{\partial t} G(t,s,0,s')u_0(s')\,ds'\right.$$

$$- c^2 \int_0^t [u(t',s')\frac{\partial}{\partial s'}G(t,s,t',s') - G(t,s,t',s')\frac{\partial}{\partial s'}u(t',s')]\Bigg|_0^1 dt'$$

$$+ \int_0^t dt' \int_0^1 ds' G(t,s,t',s')w(t',s') \Bigg\} \tag{3.37}$$

Interchanging the expectation and integration operators and using (3.11), (3.12), (3.24), and (3.25) one obtains

$$\mu(t,s) = \int_0^1 G(t,s,0,s')\mu_1(s')\, ds' + \int_0^1 \frac{\partial}{\partial t} G(t,s,0,s')\mu_0(s')\, ds' \tag{3.38}$$

where the fact that $E\{w(t,s)\} = 0$, $E\{v_0(t)\} = 0$, and $E\{v_1(t)\} = 0$ has also been used. Now the covariance $\bar{p}_{11}(t_1,s_1,t_2,s_2)$ for t_1, $t_2 \geq 0$ and $0 \leq s_1$, $s_2 \leq 1$, is to be determined by the use of (2.31). It is assumed that $w(t,s)$, $v_0(t)$, and $v_1(t)$ are statistically uncorrelated with each other and also independent of the initial conditions, then the covariance may be written as

$$\bar{p}_{11}(t_1,s_1,t_2,s_2) = \mathrm{Cov}\{u(t_1,s_1),\ u(t_2,s_2)\} \tag{3.39}$$

or

$$\bar{p}_{11}(t_1,s_1,t_2,s_2) \triangleq \int_0^1 ds_1' \int_0^1 ds_2' G_{t_1}(t_1,s_1,0,s_1')[\bar{p}_0^{11}(s_1',s_2')G_{t_2}(t_2,s_2,0,s_2')$$

$$+ \bar{p}_0^{12}(s_1',s_2')G(t_2,s_2,0,s_2')] + \int_0^1 ds_1' \int_0^1 ds_2' G(t_1,s_1,0,s_1').$$

$$[\bar{p}^{21}(s_1',s_2')G_{t_2}(t_2,s_2,0,s_2') + \bar{p}_0^{22}(s_1',s_2')G(t_2,s_2,0,s_2')]$$

$$+ c^4 \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' [\; G(t_1,s_1,t_1',0)\Big(q_{00}(t_1',t_2')G(t_2,s_2,t_2',0)$$

$$+ q_{01}(t_1',t_2')G(t_2,s_2,t_2',1)\Big)$$

$$+ G(t_1,s_1,t_1',1)\Big(q_{10}(t_1',t_2')G(t_2,s_2,t_2',0)$$

$$+ q_{11}(t_1',t_2')G(t_2,s_2,t_2',1)\Big) \;]$$

$$+ \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \int_0^1 ds_1' \int_0^1 ds_2' G(t_1,s_1,t_1',s_1')Q(t_1',s_1',t_2',s_2')G(t_2,s_2,t_2',s_2') \tag{3.40}$$

where the notation

$$G_t(t,s,0,s') = \frac{\partial}{\partial t} G(t,s,0,s') \tag{3.41}$$

and

$$Q_b(t_1,t_2) = \begin{bmatrix} q_{00}(t_1,t_2) & q_{01}(t_1,t_2) \\ q_{10}(t_1,t_2) & q_{11}(t_1,t_2) \end{bmatrix} \tag{3.42}$$

has been used.

In many physical situations such as the propogation of an electromagnetic wave or acoustic wave, temporal periodic processes are of interest [46]. In other words, the temporal variation of field is known as a random periodic process while the spatial variation is not certain due to the random property of the medium. Such problems may be easily described as a random one-point boundary-value problem resulting from the exchange of independent parameters in the random initial-value problem, i.e.,

$$\frac{\partial^2}{\partial s^2}u(t,s) = \frac{1}{c^2}\,\frac{{}^2}{t^2}u(t,s) + \frac{1}{c^2}\,w(t,s)$$

$$-\infty < t < \infty \quad ; \quad s \geq 0 \tag{3.43}$$

$$E\{u(t,0)\} = E\{u_0(t)\} = \mu_0(t) \qquad -\infty < t < \infty \tag{3.44}$$

$$E\left\{\frac{\partial}{\partial t}\,u(t,s)\Big|_{s=0}\right\} = E\{u_1(t)\} = \mu_1(t) \qquad -\infty < t < \infty \tag{3.45}$$

$$E\left\{\begin{bmatrix} u(t,0) - \mu_0(t) \\ \frac{\partial}{\partial s}\,u(t,s)\Big|_{s=0} - \mu_1(t)\end{bmatrix}\left[u(t,0) - \mu_0(t)\;\frac{\partial}{\partial s}\,u(t,s)\Big|_{s=0} - \mu_1(t)\right]\right\}$$

$$= \bar{P}_b(t-t') \qquad -\infty < t < \infty \tag{3.46}$$

where $\bar{P}_b(t-t')$ denotes a periodic covariance, (i.e., $\bar{P}_b(t-t') = \bar{P}_b(t-t' + T)$) of the field at the boundary $s = 0$. The input, $w(t,s)$ is described by (3.2) and (3.3). By exchanging independent parameters, t and s, in the initial-value problem described by (3.20) to (3.28), the moment equation is obtained as

$$\mu(t,s) = \frac{1}{2}\left[\mu_0\left(t + \frac{s}{c}\right) + \mu_0\left(t - \frac{s}{c}\right)\right] + \frac{c}{2}\int_{t-\frac{s}{c}}^{t+\frac{s}{c}} \mu_1(t')\, dt' \qquad (3.47)$$

and the covariance equation becomes

$$\begin{aligned}
\bar{p}_{11}(t_1,s_1,t_2,s_2) = \frac{1}{4}\Big[&\bar{p}_b^{11}\left(t_1 + \frac{s_1}{c}, t_2 + \frac{s_2}{c}\right) + \bar{p}_b^{11}\left(t_1 - \frac{s_1}{c}, t_2 - \frac{s_2}{c}\right) \\
&+ \bar{p}_b^{11}\left(t_1 - \frac{s_1}{c}, t_2 + \frac{s_2}{c}\right) + \bar{p}_b^{11}\left(t_1 + \frac{s_1}{c}, t_2 - \frac{s_2}{c}\right) \\
&+ c\int_{t_2-s_2/c}^{t_2+s_2/c} \left[\bar{p}_b^{12}\left(t_1 + \frac{s_1}{c}, t_2'\right) + \bar{p}_b^{12}\left(t_1 - \frac{s_1}{c}, t_2'\right)\right] dt_2' \\
&+ c\int_{t_1-s_1/c}^{t_1+s_1/c} \left[\bar{p}_b^{21}\left(t_1', t_2 + \frac{s_2}{c}\right) + \bar{p}_b^{21}\left(t_1', t_2 + \frac{s_2}{c}\right)\right] dt' \\
&+ c^2\int_{t_1-s_1/c}^{t_1+s_1/c} dt_1' \int_{t_2-s_2/c}^{t_2+s_2/c} dt_2'\, \bar{p}_b^{22}(t_1',t_2') \\
&+ c^2\int_0^{s_1} ds_1' \int_0^{s_2} ds_2' \int_{t_1-\frac{s_1-s_1'}{c}}^{t_1+\frac{s_1-s_1'}{c}} dt_1' \int_{t_2-\frac{s_2-s_2'}{c}}^{t_2+\frac{s_2-s_2'}{c}} dt_2'\, Q(t_1',s_1',t_2',s_2')\ \Big]
\end{aligned} \qquad (3.48)$$

The initial-value and initial-boundary-value problems formulated in this section will be represented in terms of temporal modes while the one-point boundary-value problem for a class of wave processes will be decomposed into a set of spatial modes. The appropriate state estimation algorithms will then be discussed, respectively.

3.3 Modal Representation of Random Field

In Section 2.3, the optimal series representation of stochastic processes has been reviewed. In view of the analytical tools available for the solution method of deterministic D.P. system such as Green's function and eigenfunction expansion methods, and the inherent computational difficulty involved in the treatment of random D.P. systems, it is believed that stochastic modal analysis can provide an elegant and computationally feasible approach to be used in the state estimation problem for D.P. systems. Fundamental to this study is the notion that the modal representation of a random field u(t,s), a sample realization of random functions in $L^2(0,1) \times \Omega$ can be achieved by means of orthogonal projections in Hilbert space [24], [47]. This is an extension of Karhunen-Loeve expansion of a random process to the study of random fields. Such an extension gives not only a set of statistically uncorrelated ordinary random differential equations for a random D.P. system, but also the optimum finite modal representation of the system in the sense of minimizing the $L^2(0,1) \times \Omega$ norm of error, the mean square error, for every fixed $t \geq 0$.

Let u(t,s) be a real valued random field in $L^2(0,1) \times \Omega$ that is,

$$E \left\{ \int_0^1 u^2(t,s) \, ds \right\} , \qquad 0 \leq s \leq 1 \quad ; \; t \geq 0 \tag{3.49}$$

exists and is finite for every fixed $t \geq 0$, and let $\{\phi_j(s)\}$ be a complete orthonormal set of elements of $L^2(0,1)$, so that

$$<\phi_j, \phi_k> = \int_0^1 \phi_j(s)\phi_k(s)\, ds = \delta_{jk} \qquad (3.50)$$

Then for every fixed value of time t and every fixed realization $\omega \in \Omega$, the random field u(t,s) can be treated as functions of s only, and the Fourier coefficients of u(t,s) with respect to the orthonormal system $\{\phi_j(s)\}$ will be stochastic processes with the temporal parameter t. The n-dimensional representation for a particular realization u(t,s) will take the form

$$u(t,s) \simeq u^*(t,s) = \sum_{j=1}^{n} x_j(t)\phi_j(s) \qquad 0 \leq s \leq 1 \qquad (3.51)$$

for every fixed t. If u(t,s) is a deterministic quantity, the equality in (3.51) is to imply the mean square convergence in $L^2(0,1)$, i.e.,

$$\lim_{n\to\infty} \left\| u(t,s) - \sum_{j=1}^{n} x_j(t)\phi_j(s) \right\|^2 \longrightarrow 0 \qquad 0 \leq s \leq 1 \qquad (3.52)$$

for every fixed $t \geq 0$, and the choice of the complete orthonormal system $\{\phi_j(s)\}$ is not unique. However, for the case of a random field u(t,s) the equality in (3.51) has the meaning

$$u(t,s) = \underset{n \to \infty}{\text{l.i.m.}} \sum_{j=1}^{n} x_j(t)\phi_j(s) \qquad 0 \leq s \leq 1 \qquad (3.53)$$

or

$$\lim_{n \to \infty} E\left\{\left\| u(t,s) - \sum_{j=1}^{n} x_j(t)\phi_j(s)\right\|^2\right\} \to 0 \tag{3.54}$$

for every fixed $t \geq 0$, and it is possible to find the optimum coordinate system $\{\phi_j(s)\}$ on the basis of the statistical property of the given process. Taking the steps analogous to those shown in Section 2.3, one obtains an expression for the error criterion:

$$\begin{aligned} E\left\{\|u(t,s) - u^*(t,s)\|^2\right\} &= E\left\{ \langle u(t,s) \ , \ u(t,s)\rangle \right\} \\ &+ E\left\{ \sum_{j=1}^{n} |\langle u(t,s) \ , \ \phi_j(s)\rangle - x_j(t)|^2 \right\} \\ &- E\left\{ \sum_{j=1}^{n} |\langle u(t,s) \ , \ \phi_j(s)\rangle|^2 \right\} \\ & 0 \leq s \leq 1 \end{aligned} \tag{3.55}$$

for every fixed $t \geq 0$. By making

$$\begin{gathered} x_j(t) = \langle u(t,s) \ , \ \phi_j(s)\rangle \\ 0 \leq s \leq 1 \quad ; \quad j = 1,2, \ldots n \end{gathered} \tag{3.56}$$

for every fixed $t \geq 0$, the second term on the right achieves its minimum value for each realization. Now the last term on the right

$$E\left\{\sum_{j=1}^{n}\|\langle u(t,s),\phi_j(s)\rangle\|^2\right\} = \sum_{j=1}^{n}\int_0^1\int_0^1 E\{u(t,s)u(t',s')\}\phi_j(s)\phi_j(s')ds\,ds' \tag{3.57}$$

for $0 \leq s\ s' \leq 1$ and every fixed $t,t' \geq 0$, is to be maximized by the choice of an appropriate set of orthonormal functions $\{\phi_j(s)\}$. From the previous discussion of Karhunen-Loeve expansions, this can be achieved by taking the n eigenfunctions of the homogeneous integral equation

$$\int_0^1 \bar{p}(t,s,t',s')\phi_j(s')\,ds' = p(t,t')\phi_j(s) \qquad 0 \leq s \leq 1 \tag{3.58}$$

corresponding to the n largest eigenvalues $p(t,t')$ for every fixed $t,t' \geq 0$ where $\bar{p}(t,s,t',s') = E\,\{u(t,s)\,u(t',s')\}$. An easy way to show this is to apply the Schwartz's inequality,

$$\left[\int_0^1 f(s)g(s)\,ds\right]^2 \leq \int_0^1 f^2(s)ds \int_0^1 g^2(s)ds \tag{3.59}$$

where $f(s)$ and $g(s)$ are in $L^2(0,1)$, to each term on the right of (3.57). Let $f(s) = \int_0^1 \bar{p}(t,s,t',s')\,\phi_j(s')\,ds'$ and $g(s) = \phi_j(s)$ for every fixed $t,\ t' \geq 0$ and $0 \leq s \leq 1$. Since the Schwarz's inequality (3.59) can be rewritten as

$$\int_0^1 f(s)g(s)ds \leq ||f(s)||\,||g(s)|| \tag{3.60}$$

one can write

$$\int_0^1 ds\left[\int_0^1 \bar{p}(t,s,t',s')\phi_j(s')ds'\right]\phi_j(s)\ ds \quad \leq$$

$$\|\langle p(t,s,t',s')\ ,\ \phi_j(s)\rangle\|\cdot\|\phi_j(s)\| \tag{3.61}$$

for every fixed $t,t' \geq 0$ and $0 \leq s \leq 1$. In order to have equality in (3.61) one need $f(s) = k\,g(s)$ where k is a constant. For (3.57) this implies that

$$\int_0^1 p(t,s,t's')\phi_j(s')ds' = p(t,t')\phi_j(s) \qquad \begin{array}{l} 0 \leq s \leq 1\ ; \\ j = 1,2,\ .\ .\ . \end{array} \tag{3.62}$$

which gives the relationship to be satisfied by the optimal set of basis functions $\{\phi_j(s)\}$ for every fixed $t,t' \geq 0$. Therefore, the optimum n-dimensional subspace of $L^2(0,1)$ for representing realizations of a random field over the interval, $0 \leq s \leq 1$, is spanned by the n eigen-functions of the integral equation (3.62). The properties of the integral equation discussed in Section 2.3 are also true for every fixed $t \geq 0$ in (3.62). It follows that

$$\bar{p}(t,s,t',s') = \sum_{j=1}^{\infty} p_j(t,t')\,\phi_j(s)\,\phi_j(s') \tag{3.63}$$

and (3.55) becomes

$$E\left\{\|u(t,s) - u^*(t,s)\|^2\right\} = \sum_{j=1}^{\infty} p_j(t,t')<\phi_j(s),\phi_j(s)>$$

$$- \sum_{j=1}^{n} p_j(t,t')$$

$$= \sum_{n+1}^{\infty} p_j(t,t') \qquad (3.64)$$

for every fixed $t,t' \geq 0$ and $0 \leq s \leq 1$. The coefficients $x_j(t)$ in the expansion

$$u^*(t,s) = \sum_{j=1}^{n} x_j(t)\phi_j(s) \qquad 0 \leq s \leq 1 \qquad (3.65)$$

are also uncorrelated random variables for every fixed $t \geq 0$, since

$$E\left\{x_i(t)x_j(t')\right\} = E\left\{<u(t,s),\phi_i(s)><u(t',s'),\phi_j(s')>\right\}$$

$$= \int_0^1\int_0^1 \bar{p}(t,s,t',s')\phi_i(s)\phi_j(s')\, ds\, ds'$$

$$= p_j(t,t')\ \delta_{ij}$$

$$0 \leq s \leq 1 \qquad (3.66)$$

where (3.50) and (3.63) are utilized.

For its use in the following study, the modal expansion of a white random field defined by

$$E\{w(t,s)\} = 0 \tag{3.67}$$

$$E\{w(t,s)w(t',s')\} = q(t,t')\,\delta(s-s') \tag{3.68}$$

is of interest. To find the set of optimal basis functions $\{\phi_j(s)\}$ the integral equation (3.62) with the kernel given by (3.68) is to be solved:

$$\int_0^1 q(t,t')\,\delta(s-s')\phi_j(s')\,ds' = p_j(t,t')\phi_j(s)$$

From the sifting property of the δ function, the equation is satisfied for any $\phi_j(s)$ with $p_j(t,t') = q(t,t')$. Thus an arbitrary set of orthonormal functions is suitable for the finite dimensional representation of a white random field

$$w(t,s) \simeq w^*(t,s) = \sum_{j=1}^{n} w_j(t)\,\phi_j(s) \tag{3.69}$$

The reason for the nonuniqueness of the orthonormal basis function is that the impulse kernel is not square-integrable. Thus there is no sequence for the random field $w^*(t,s)$ which is mean square convergent for each t and s and can converge to a white noise. However it should not be forgotten that a white noise is not a physical process, but merely an abstraction of a physical phenomena. Therefore, the orthonormal expansion of white noise is often considered, even though the series is not convergent [23].

3.4 Modal Representation of Random Diffusion or Heat Equation

In the last section, it was shown that the optimal set of basis function to decompose a random field can be found by solving a homogeneous integral equation endowed with the covariance kernel of the field. However the solution of the integral equation for every fixed temporal parameters t and t' will provide an ever changing set of optimal basis function for t, t' $\geq$ 0 even when the covariance kernel is known. Fortunately, from the formulation of the random diffusion or heat problems in Section 3.2.1 the covariance propagation is obtained in terms of Green's function for the system. For the initial-boundary-value problem it was known that the Green's function may be expressed in series form. While the set of basis function for the deterministic problem is obtained by solving an eigenfunction problem associated with the Sturm-Lioville's differential equation having a homogeneous boundary condition [44], for the random problem it is to be found by solving the eigenfunction problem of the equivalent integral equation.

The modal representation of a random heat process is found as follows: By combining (3.56) and (3.57) in (3.55), the mean squared norm of error in approximating the random field can be rewritten as

$$E\left\{\|u(t,s) - u^*(t,s)\|^2\right\} = E\left\{<u(t,s),u(t,s)>\right\}$$

$$- \sum_{j=1}^{n} \int_0^1\!\!\int_0^1 E\{u(t_1,s_1)u(t_2,s_2)\}\,\phi_j(s_1)\phi_j(s_2)ds_1 ds_2$$

$$0 \leq s\ ,s\ \leq 1 \quad ; \quad t \geq 0 \tag{3.70}$$

where s_1, and s_2 are equivalent to s, and s' in (3.57). Now to maximize the last term on the right with respect to each orthonormal basis function $\phi_j(s)$, under the constraints given by the random initial-boundary-value problem defined in Section 3.2.1, it is natural to substitute the covariance propagation given by (3.19) into (3.70). Since the Green's function may be expressed in terms of the orthonomal series expansion of (3.26), one obtains the result that the last term of (3.70) is of the form

$$\sum_{j=1}^{n} \int_0^1\int_0^1 E\left\{u(t_1,s_1)u(t_2,s_2)\right\} \phi_j(s_1)\,\phi_j(s_2)ds_1 ds_2 =$$

$$\sum_{j=1}^{n} \int_0^1 ds_1 \int_0^1 ds_2 \left[\int_0^1 ds_1' \int_0^1 ds_2' \left(\sum_{i=1}^{\infty} e^{-k\lambda_i t_1} \phi_i(s_1)\phi_i(s_1')\right)\bar{p}_0(s_1',s_2') \cdot \right.$$

$$\left(\sum_{m=1}^{\infty} e^{-k\lambda_m t_2} \phi_m(s_2)\phi_m(s_2')\right)$$

$$+\ k^2\int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \left[\left(\sum_{i=1}^{\infty} e^{-k\lambda_i(t_1-t_1')} \phi_i(s_1)\phi_i(0)\right)\left(q_{00}(t_1',t_2')\sum_{m=1}^{\infty} e^{-k\lambda_m(t_2-t_2')} \phi_m(s_2)\phi_m(0)\right.\right.$$

$$\left. +\ q_{01}(t_1',t_2')\sum_{m=1}^{\infty} e^{-k\lambda_m(t_2-t_2')} \phi_m(s_2)\phi_m(1)\right)$$

$$+ \left(\sum_{i=1}^{n} e^{-k\lambda_i(t_1-t_1')}\phi_i(s_1)\phi_i(1)\right)\left(q_{10}(t_1',t_2')\sum_{m=1}^{\infty} e^{-k\lambda_m(t_2-t_2')}\phi_m(s_2)\phi_m(0)\right.$$

$$\left.\left. + q_{11}(t_1',t_2')\sum_{m=1}^{\infty} e^{-k\lambda_m(t_2-t_2')}\phi_m(s_2)\phi_m(1)\right)\right]$$

$$+ \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \int_0^1 ds_1' \int_0^1 ds_2' \left(\sum_{i=1}^{\infty} e^{-k\lambda_i(t_1-t_1')}\phi_i(s_1)\phi_i(s_1')\right) \cdot$$

$$\left. Q(t_1',s_1',t_2',s_2')\left(\sum_{m=1}^{\infty} e^{-k\lambda_m(t_2-t_2')}\phi_m(s_2)\phi_m(s_2')\right)\right]\phi_j(s_1)\phi_j(s_2) \tag{3.71}$$

The expression on the right can be simplified by the assumption that $\{\phi_j(s)\}$ is an orthonormal set, and using the white noise property of $v_0(t)$, $v_1(t)$, and $w(t,s)$. The resulting expression becomes

$$\sum_{j=1}^{n} \int_0^1 ds_1 \int_0^1 ds_2 \; E\left\{u(t_1,s_1)u(t_2,s_2)\right\}\phi_j(s_1)\phi_j(s_2)$$

$$= \sum_{j=1}^{n} \left[e^{-k\lambda_j(t_1+t_2)}\int_0^1 ds_1' \int_0^1 ds_2' \; \bar{p}_0(s_1',s_2')\phi_j(s_1')\phi_j(s_2')\right.$$

$$+ k^2 \int_0^{\min\{t_1,t_2\}} dt' \; e^{-k\lambda_j(t_1+t_2-2t')}\left[\phi_j^2(0)q_{00}(t') + \right.$$

$$\phi_j(1)\phi_j(0)q_{01}(t') + \phi_j(1)\phi_j(0)q_{10}(t') + \phi_j^2(1)q_{11}(t')]$$

$$+ \int_0^{\min\{t_1,t_2\}} dt' \; e^{-k\lambda_j(t_1+t_2-2t')} \Big[\Big[\int_0^1 ds_1' \int_0^1 ds_2' Q(t')\,\delta(s_1'-s_2')\,\phi_j(s_1')\,\phi_j(s_2')\Big] \Big]$$

(3.72)

where special attention is directed in selecting the order of double integration and in using the sifting property of delta functions [49]. To maximize the above expression for every fixed t_1 and t_2, it is convenient to restrict attention to the case in which $t_1 = t_2 = t$, so that the term on the right is given by

$$\sum_{j=1}^{n} \Big[e^{-2k\lambda_j t} \int_0^1 ds_1' \int_0^1 ds_2' \; \bar{p}_0(s_1',s_2')\,\phi_j(s_1')\,\phi_j(s_2')$$

$$+ k^2 \int_0^t dt' \; e^{-2k\lambda_j(t-t')} [\phi_j^2(0)q_{00}(t') + \phi_j(0)\,\phi_j(1)q_{01}(t')$$

$$+ \phi_j(1)\,\phi_j(0)q_{10}(t') + \phi_j^2(1)q_{11}(t')]$$

$$+ \int_0^t dt' \; e^{-2k\lambda_j(t-t')} \Big[\int_0^1 ds_1' \int_0^1 ds_2' \; Q(t') \;\; \delta(s_1'-s_2')\,\phi_j(s_1')\,\phi_j(s_2')\Big] \Big]$$

In maximizing the expression with respect to $\{\phi_j(s)\}$ for every fixed $t \geq 0$, the second term is independent of $\{\phi_j(s)\}$, and the expression in the bracket of the third term looks familiar to one from Section 3.3, where the expansion of white noise was considered. It was learned that any set of spatial coordinate function may be used in decomposing the forcing function. Although the second term contains the set $\{\phi_j(0), \phi_j(1)\}$, it does not affect the choice of $\{\phi_j(s)\}$ since the boundary conditions are to be satisfied. The result is most important and allows one to consider only the first term which has to do with the statistics of the initial conditions. Hence, the only term to be maximized is

$$\sum_{j=1}^{n} e^{-k\lambda_j t} \int_0^1 ds_1' \int_0^1 ds_2' \, \bar{p}_0(s_1', s_2') \phi_j(s_1') \phi_j(s_2') \tag{3.73}$$

which has already been investigated in studying the series representation of random processes in Section 2.3. It was required that the set of orthonormal coordinate function $\{\phi_j(s)\}$ be the eigenfunction of the homogeneous integral equation:

$$p_j(0)\phi_j(s) = \int_0^1 p_0(s,s')\phi_j(s') \, ds'$$

$$0 \leq s \leq 1 \quad ; \quad j = 1, 2, \ldots n \tag{3.74}$$

where $p_j(0)$ is the corresponding eigenvalue, representing the amount of energy associated with each coordinate function $\phi_j(s)$. It is noted that the kernel function $\bar{p}_0(s,s')$ may be expressed uniformly:

$$\bar{p}_0(s,s') = \sum_{j=1}^{\infty} p_j(0)\,\phi_j(s)\,\phi_j(s') \qquad (3.75)$$

Substituting this result into (3.73), one obtains

$$p^*(t) = \sum_{j=1}^{n} \Bigg(p_j(0)\, e^{-2k\lambda_j t} + \int_0^t dt'\, e^{-2k\lambda_j(t-t')} [\phi_j^2(0) q_{00}(t') + \phi_j(0)\phi_j(1) \cdot (q_{01}(t') + q_{10}(t')) + \phi_j^2(1) q_{11}(t)] + \int_0^t dt'\, e^{-2k\lambda_j(t-t')} Q_j(t) \Bigg) \qquad (3.76)$$

where the term $Q_j(t)$ is defined as

$$Q_j(t) = \int_0^1 Q(t,s)\,\phi_j(s)\, ds \qquad (3.77)$$

The expression $p^*(t)$ defines the sum of the variances for each random coefficient process $\{x_j(t)\}$, resulting from the n-modes approximation.

Now, the modal expression of the random initial-boundary-value problem is obtained by substituting the Green's function solution form of $u(t,s)$ into (3.56), which becomes

$$x_j(t) = \int_0^1 ds\phi_j(s)\left[\int_0^1 u(0,s')\sum_{i=1}^{\infty} e^{-k\lambda_i t}\phi_i(s)\phi_i(s')\,ds'\right.$$

$$+ k\int_0^t\left(v_0(t')\sum_{i=1}^{\infty} e^{-k\lambda_i t}\phi_i(s)\phi_i(0) + v_1(t')\sum_{i=1}^{\infty} e^{-k\lambda_i t}\phi_i(s)\phi_i(1)\right)dt'$$

$$\left. + \int_0^t dt'\int_0^1 ds'\, w(t',s')\sum_{i=1}^{\infty} e^{-k\lambda_i(t-t')}\phi_i(s)\phi_i(s')\right] \qquad (3.79)$$

where the series expansion of Green's function (2.14) and the properties of Green's function, (2.20) to (2.21), are utilized. Performing integration over the parameter s one obtains

$$x_j(t) = e^{-k\lambda_j t}x_{0j} + \int_0^t e^{-k\lambda_j(t-t')}[\, w_j(t') + k\phi_j(0)v_0(t')$$

$$+ k\phi_j(1)v_1(t')\,]\,dt'$$

$$t \geq 0 \quad ; \quad j = 1, 2, \ldots n \qquad (3.80)$$

where

$$x_{0j} = \int_0^1 u(0,s')\phi_j(s')\,ds' \qquad (3.81)$$

and

$$w_j(t') = \int_0^1 w(t',s')\phi_j(s')\, ds' \tag{3.82}$$

are defined. It is apparent that the resulting expression obtained in (3.80) represent a familiar solution form

$$x(t) = \Phi(t-t_0)x_0 + \int_{t_0}^{t} \Phi(t-t')D\bar{w}(t')\, dt' \tag{3.83}$$

of the state transition matrix satisfying

$$\dot{\Phi}(t-t_0') = A\,\Phi(t-t_0) \tag{3.84}$$

$$\Phi(t_0) = I \tag{3.85}$$

for a linear lumped parameter system defined by

$$\dot{x}(t) = A\,x(t) + D\,\bar{w}(t) \tag{3.86}$$

$$x(t_0) = x_0 \tag{3.87}$$

where an analogy is made as follows:

$$A = \mathrm{diag}\,[-k\lambda_1, -k\lambda_2, \ldots -k\lambda_n] \tag{3.88}$$

$$\Phi(t-t') = \mathrm{diag}\,[\, e^{-k\lambda_1(t-t')}, e^{-k\lambda_2(t-t')}, \ldots e^{-k\lambda_n(t-t')}] \tag{3.89}$$

$$D = \begin{bmatrix} k\phi_1(0) & k\phi_1(1) & 1 & 0 & . & . & . & 0 \\ k\phi_2(0) & k\phi_2(1) & 0 & 1 & . & . & . & 0 \\ \vdots & \vdots & \vdots & \vdots & & & & \vdots \\ k\phi_n(0) & k\phi_n(1) & 0 & 0 & . & . & . & 1 \end{bmatrix} \tag{3.90}$$

and

$$\bar{w}(t) = [v_0(t),\ v_1(t),\ w_1(t),\ w_2(t),\ .\ .\ .\ w_n(t)]^T \tag{3.91}$$

This is due to the fact that the Green's function, impulse response, of the random D.P. system has been expanded by a set of optimum spatial coordinate function, yielding infinite dimensional mode propagation in the temporal parameter t. Since u(0,s) and w(t,s) are random quantities and $\{\phi_j(s)\}$ is deterministic, x_{0j} and $w_j(t)$ are characterized by

$$E\{x_{0j}\} = \int_0^1 E\{u(0,s')\}\phi_j(s')\ ds' = 0$$

$$E\{w_j(t)\} = \int_0^1 E\{w(t',s')\}\phi_j(s')\ ds' = 0 \tag{3.92}$$

$$E\{x_{0j}x_{0k}\} = \int_0^1 ds \int_0^1 ds'\ E\{u(0,s)u(0,s')\}\phi_j(s)\phi_k(s')$$

$$= \int_0^1 ds \int_0^1 ds'\ \bar{p}_0(s,s')\phi_j(s)\phi_k(s')$$

$$= \int_0^1 ds \int_0^1 ds' \left(\sum_{i=1}^{\infty} p_i(0) \phi_i(s) \phi_i(s') \right) \phi_j(s) \phi_k(s')$$

$$= \; p_j(0) \; \delta_{jk}$$

and

$$E\{w_j(t) w_k(t)\} = \int_0^1 ds \int_0^1 ds' \; E\{w(t,s) w(t',s')\} \phi_j(s) \phi_k(s')$$

$$= \int_0^1 ds \int_0^1 ds' \; Q(t,t') \delta(s-s') \phi_j(s) \phi_k(s')$$

$$= \; Q(t) \; \delta(t-t') \; \delta_{jk} \tag{3.93}$$

In vector form, that is,

$$E\{x(0)\} = 0 \tag{3.94}$$

$$E\{\bar{w}(t)\} = 0 \tag{3.95}$$

$$E\{\bar{w}(t) \bar{w}^T(t')\} = \begin{bmatrix} q_{00}(t) & q_{01}(t) & & & \\ q_{10}(t) & q_{11}(t) & & 0 & \\ & & Q(t) & & 0 \\ & 0 & & \ddots & \\ & & 0 & & Q(t) \end{bmatrix} \tag{3.96}$$

Thus, it is possible to express the optimum n dimensional modal representation u*(t,s) of the random initial-boundary-value heat problem given by (3.4), (3.5), (3.6), (3.11), (3.12), (3.13) and (3.14), namely, as

$$u^*(t,s) = \sum_{j=1}^{n} x_j(t)\phi_j(s) \tag{3.97}$$

where $x_j(t)$ is the solution of the system of ordinary random differential equations defined by (3.83) to (3.96), and $\{\phi_j(s)\}$ is the optimum set of orthonormal basis function obtained by solving the homogeneous integral equation (3.74). It is worthwhile to note that the well-posedness assumption of the initial-boundary-value problem is sufficient to justify that the set of eigenfunctions $\{\phi_j(s)\}$ of the integral equation allows the Green's function to satisfy the boundary conditions given by (3.17) and (3.18). This follows from the fact that the solution of the Sturm Liouville problem

$$\frac{d^2}{ds^2}\,\phi(s) + \lambda\phi(s) = 0 \tag{3.98}$$

subject to a pair of homogeneous boundary conditions

$$\frac{d}{ds'}\,\phi(s')\bigg|_{s'=0} + b_0\,\phi(0) = 0 \tag{3.99}$$

$$\frac{d}{ds'}\,\phi(s')\bigg|_{s'=1} + b_1\,\phi(1) = 0 \tag{3.100}$$

can be written in terms of the homogeneous integral equation [27],

$$\phi(s) = \lambda \int_0^1 G(s,s')\phi(s')\, ds' \tag{3.101}$$

where the Green's function $G(s,s')$ is a symmetric kernel and b_0 and b_1 are positive real constants. In view of the relationship between (3.98) and (3.100) it follows that the well-posedness of the initial-boundary-value problem implies $G(s,s') = \bar{p}_0(s,s')$.

When the spatial region of interest is infinite, $-\infty<s<\infty$, similar analysis may be carried out for a class of random initial-value problem, (3.4) to (3.13), which exhibits a periodic random process at t=0, i.e.,

$$\bar{p}_0(\sigma) = \bar{p}_0(\sigma+1) \tag{3.102}$$

where $\bar{p}_0(\cdot)$ is the autocovariance with a period and $\sigma = s-s'$. In this case the temporal mode representation can be derived by restricting one's attention to a spatial period, $0\leq s\leq 1$. The resulting system of ordinary random differential equations will be exactly the same as (3.83) through (3.97) except for the absence of the boundary terms. Consequently, the set of equations consists of statistically independent modes.

Illustrative Example:

The following example illustrates the procedure developed so far in this section for the modal representation of a random heat problem. Consider a right cylinder of lengh one, whose lateral surface is insulated and one of the basis is disturbed at a random temperature. It is subjected to controlled heating via embedded elements. The random field u(t,s) represents the temperature distribution along the length of the cylinder, having a spatial wiener process at t=0. Imperfect insulation and nonuniform heating produce random disturbances in the system.

Let the system model be given by:

$$\frac{\partial}{\partial t} u(t,s) = k\frac{\partial^2}{\partial s^2}u(t,s) + w(t,s)$$

$$0 \leq s \leq 1 \quad ; \quad t \geq 0 \tag{3.103}$$

where k is the constant diffusivity of the material and w(t,s) is a white Gaussian random field with the statistics by

$$E\{w(t,s)\} = 0 \tag{3.104}$$

$$E\{w(t,s)w(t',s')\} = q \quad (t-t',s-s') \tag{3.105}$$

$$0 \leq s \leq 1 \quad ; \quad t \geq 0$$

The initial conditions are Gaussian with statistics given by

$$E\{u(0,s)\} = 0 \tag{3.106}$$

$$E\{u(0,s)u(0,s')\} = \bar{p}_0 \min(s,s')$$

$$0 \leq s,s' \leq 1 \tag{3.107}$$

Let the boundary conditions be

$$E\{v_0(t)\} = E\{v_1(t)\} = 0 \tag{3.108}$$

$$E\left\{\begin{bmatrix} v_0(t) \\ v_1(t) \end{bmatrix}[v_0(t') \ \ v_1(t)]\right\} = \begin{bmatrix} q_{00} & 0 \\ 0 & q_{11} \end{bmatrix}\delta(t-t') \tag{3.109}$$

In order to find the modal representation of the problem, the first step is to obtain the set of basis function in space. The homogeneous integral equation

$$p_j(0)\phi_j(s) = \bar{p}_0\int_0^1 \min(s,s')\phi_j(s')ds' \quad j = 1,2... \tag{3.110}$$

can be solved rather easily by finding the corresponding differential equation, solving it, and substituting it back into the integral equation [23] or by state-variable methods [26]. It is found that the Karhunen-Loeve expansion has the eigenfunctions and eigenvalues of the form

$$\phi_j(s) = \sqrt{2}\, \sin(j + \tfrac{1}{2})\, \pi s \tag{3.111}$$

and

$$p_j(0) = \bar{p}_0/(j + \tfrac{1}{2})^2\pi^2$$

$$0 \le s \le 1 \ ; \ j=1,2,... \tag{3.112}$$

Then the series expansion for the initial state is

$$u(0,s) = \sum_{j=1}^{\infty} x_j(0)\sqrt{2}\,\sin(j + 1/2)\pi s \tag{3.113}$$

where

$$x_j(0) = \int_0^1 u(0,s)\sqrt{2}\,\sin(j + 1/2)\pi s\, ds \tag{3.114}$$

The Sturm-Liouville differential equation

$$\frac{d^2}{ds^2}\phi_j(s) = -\lambda_j\phi_j(s) \tag{3.115}$$

can be satisfied for each basis function when the eigenvalues are

$$\lambda_j = (j + 1/2)^2\pi^2 \tag{3.116}$$

From (3.83) the modal representation of the system model is

$$\frac{d}{dt}\,x_j(t) = -k\lambda_j x_j(t) - \sqrt{2}kv_1(t) + w_j(t) \tag{3.117}$$

where the initial conditions and its statistics are given by

$$E\,\{x_j(0)\} = \int_0^1 E\{u(0,s)\}\phi_j(s)\, ds = 0 \tag{3.118}$$

$$E\{x_j(0)x_k(0)\} = \frac{\bar{p}_0}{(j + 1/2)^2\pi^2} \delta_{jk} \tag{3.119}$$

The system disturbances are characterized by

$$E\{w_j(t)\} = 0 \tag{3.120}$$

$$E\{w_j(t)w_k(t')\} = q\,\delta(t-t')\,\delta_{jk} \tag{3.121}$$

Observe that the resulting modal representation consists of uncorrelated and statistically independent part plus correlated boundary noise term. Of course it is not difficult to see that for an initial value problem, a set of statistically independent stochastic ordinary differential equations will result. It is also noted that when the solution of the integral equation (3.74) provides only a finite number of basis functions, the complete set of eigenfunctions may be required to decompose the forcing term of the inhomogeneous problem.

In Section 4.4.1, an application of an estimation algorithm will be discussed for this particular example problem.

3.5 Modal Representation of the Random Wave Equation

It is not difficult to extend the results obtained for the heat processes to the study of random wave processes. The existence of the second derivative terms in both parameters, t and s, gives some different, feature of the problem. In many applications of deterministic wave analysis, the time harmonic assumption is made to facilitate the study. The

Helmholtz's wave equation in terms of spatial parameters results. When stochastic harmonic behaviour in time is encountered, one may need a stochastic Helmholtz's equation. Thus, in addition to the study of temporal mode representations, a Markov property of a semi-infinite space is assumed and its spatial mode representation is pursued.

3.5.1 Temporal Mode Representation

The random initial-boundary-value problem defined by (3.11) through (3.14) and (3.24) to (3.27) is considered here using an approach analogous to that taken for the heat problem. The finite mode representation of the solution field u(t,s) may be expressed as

$$u^*(t,s) = \sum_{j=1}^{n} x_{2j-1}(t)\ \phi_j(s) \qquad 0 \le s \le 1 \ ;\ t \ge 0 \quad ;\ j = 1, 2, \ldots n \tag{3.122}$$

where the set of basis functions $\{\phi_j(s)\}$ is an orthonormal set in $L^2[0,1]$. The coefficient $x_{2j-1}(t)$ for every fixed t is to be obtained by minimizing the mean square error

$$E\left\{ \|u(t,s) - u^*(t,s)\|^2 \right\} = E\left\{ <u(t,s), u(t,s)> \right\}$$

$$- \sum_{j=1}^{n} \int_0^1\!\!\int_0^1 E\{u(t_1,s_1)u(t_2,s_2)\}\,\phi_j(s_1)\ \phi_j(s_2)\ ds_1\ ds_2$$

$$0 \le s_1, s_2 \le 1 \quad ; \quad t_1, t_2 \ge 0 \tag{3.123}$$

which is rewritten from (3.70). As a result, the second term on the right is to be maximized with respect to the set $\{\phi_j(s)\}$, given the stochastic initial-boundary-value wave problem. One does not know the explicit expression of the covariance

$$E\{u(t_1,s_1)u(t_2,s_2)\} = \bar{p}(t_1,s_1,t_2,s_2) \tag{3.124}$$

but the integral equation expression, i.e., the Green's function expression, of (3.124) is found in (3.40). Since the Green's function of the initial-boundary-value problem may be found in terms of an orthonormal series expansion of (2.33) one can rewrite the last term on the right of (3.123) by combining (3.40) and (2.32):

$$\sum_{j=1}^{n} \int_0^1\int_0^1 \bar{p}(t_1,s_1,t_2,s_2)\, \phi_j(s_1)\, \phi_j(s_2)\, ds_1\, ds_2$$

$$= \sum_{j=1}^{n} \int_0^1 ds_1 \int_0^1 ds_2 \Bigg[\int_0^1 ds_1' \int_0^1 ds_2' \left(\sum_{i=1}^{\infty} \cos\omega_i ct_1 \phi_i(s_1)\phi_i(s_1') \right)$$

$$\left(p_0'(s_1',s_2') \sum_{m=1}^{\infty} \cos\omega_m ct_2 \phi_m(s_2)\phi_m(s_2') \right.$$

$$\left. + p_0^{12}(s_1',s_2') \sum_{m=1}^{\infty} \frac{\sin\omega_m ct_2}{\omega_m c}\, \phi_m(s_2)\phi_m(s_2') \right.$$

$$+ \int_0^1 ds_1' \int_0^1 ds_2' \left(\sum_{i=1}^{\infty} \frac{\sin\omega_i ct_1}{\omega_i c}\, \phi_i(s_1)\phi_i(s_1') \right)$$

$$\left(p_0^{21}(s_1',s_2') \sum_{m=1}^{\infty} \cos\omega_m ct_2\, \phi_m(s_2)\phi_m(s_2') \right.$$

$$\left. + p_0^{22}(s_1',s_2') \sum_{m=1}^{\infty} \frac{\sin\omega_m ct_2}{\omega_m c}\, \phi_m(s_2)\phi_m(s_2') \right)$$

$$+ c^4 \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \left[\sum_{i=1}^{\infty} \frac{\sin\omega_i c(t_1-t_1')}{\omega_i c}\, \phi_i(s_1)\phi_i(0) \right.$$

$$\Big(q_{00}(t_1',t_2')\sum_{m=1}^{\infty}\frac{\sin\omega_m c(t_2-t_2')}{\omega_m c}\phi_m(s_2)\phi_m(0)$$

$$+\,q_{01}(t_1',t_2')_{m=1}\frac{\sin\,\omega_m c(t_2-t_2')}{\omega_m c}\phi_m(s_2)\phi_m(1)\Big)$$

$$+\sum_{i=1}^{\infty}\frac{\sin\,\omega_i c(t_1-t_1')}{\omega_i c}\phi_i(s_1)\phi_i(1)\Big(q_{10}(t_1',t_2')_{m=1}\frac{\sin\,\omega_m c(t_2-t_2')}{\omega_m c}\phi_m(s_2)\phi_m(0)$$

$$+\,q_{11}(t_1',t_2')\sum_{m=1}^{\infty}\frac{\sin\omega_m c(t_2-t_2')}{\omega_m c}\phi_m(s_2)\phi_m(1)\Big)\Big]$$

$$+\int_0^{t_1}dt_1'\int_0^{t_2}dt_2'\int_0^{1}ds_1'\int_0^{1}ds_2'\Big(\sum_{i=1}^{\infty}\frac{\sin\,\omega_i c(t_1-t_1')}{\omega_i c}\phi_i(s)\phi_i(s'_1)\Big)\,Q(t_1',s_1',t_2',s_2')$$

$$\Big(\sum_{m=1}^{\infty}\frac{\sin\omega_m c(t_2-t_2')}{\omega_m c}\phi_m(s_2)\phi_m(s_2')\Big)\Big]\phi_j(s_1)\phi_j(s_2)$$

(3.125)

From the orthonormality of $\{\phi_j(s)\}$ and the whiteness assumption of $v_0(t)$, $v_1(t)$ and $w(t,s)$, the term on the right becomes

$$\sum_{j=1}^{n}\int_0^1 ds_1'\int_0^1 ds_2'\phi_j(s_1')\phi_j(s_2')\Big[p_0^{11}(s_1',s_2')(\cos\omega_j ct_1\;\cos\omega_j ct_2)$$

$$+(p_0^{12}(s_1',s_2')+p_0^{21}(s_1',s_2'))(\frac{1}{\omega_j c}\cos\omega_j ct_1\;\sin\omega_j ct_2)$$

$$+\,p_0^{22}(s',s')(\frac{1}{\omega_j^2c^2}\sin\omega_j ct\;\sin\omega_j ct_2)\Big]$$

$$+ c^4 \int_0^{\min\{t_1,t_2\}} \frac{1}{\omega_j^2 c^2} \sin\omega_j c(t_1-t') \sin\omega_j c(t_2-t') \Big[q_{00}(t')\phi_j^2(0) +$$

$$\Big(q_{01}(t') + q_{10}(t')\Big)\phi_j(0)\phi_j(1) + q_{11}(t')\phi_j^2(1)\Big]dt'$$

$$+ \int_0^{\min\{t_1,t_2\}} dt' \frac{1}{\omega_j^2 c^2} \sin\omega_j c(t_1-t')\sin\omega_j c(t_2-t')\Big[\int_0^1 ds_1' \int_0^1 ds_2' \; Q(t',s_1',s_2')\phi_j(s_1')\phi_j(s_2')\Big] \tag{3.126}$$

Since one is to maximize this expression for every fixed $t \geq 0$, attention is restricted to the case $t_1=t_2=t$. Then one considers the term

$$\sum_{j=1}^{n} \Big[\cos^2\omega_j ct \int_0^1 ds_1' \int_0^1 ds_2' \; p_0^{11}(s_1',s_2')\phi_j(s_1')\phi_j(s_2')$$

$$+ \frac{1}{\omega_j c} \cos\omega_j ct \; \sin\omega_j ct \int_0^1 ds_1' \int_0^1 ds_2' [p_0^{12}(s_1',s_2') + p_0^{21}(s_1',s_2')]\phi_j(s_1')\phi_j(s_2')$$

$$+ \frac{1}{\omega_j c} \sin\omega_j^2 ct \int_0^1 ds_1' \int_0^1 ds_2' \; p_0^{22}(s_1',s_2')\phi_j(s')\phi_j(s')$$

$$+ \int_0^t \frac{1}{\omega_j^2 c^2} \sin^2\omega_j c(t-t')\Big[c^4(q_{00}(t')\phi_j^2(0) + q_{01}(t')\phi_j(0)\phi_j(1) +$$

$$q_{10}(t')\phi_j(0)\phi_j(1) + q_{11}(t')\phi_j^2(1) \;)$$

$$+ \int_0^1 ds_1' \int_0^1 ds_2' \; Q(t',s_1',s_2')\phi_j(s')\phi_j(s')\Big] dt'\Big] \tag{3.127}$$

As discussed in the previous section the last term is independent in choosing $\{\phi_j(s)\}$ when the whiteness of the disturbances is assumed.

In maximizing the first three terms with repect to $\{\phi_j(s)\}$ for every fixed $t \geq 0$ the second term is cumbersome so it is necessary to examine the elements of the matrix $\bar{P}_0(s,s')$ under the assumption that

$$u(0,s) = \sum_{j=1}^{\infty} x_{2j-1}(0)\phi_j(s) \tag{3.128}$$

and

$$\frac{\partial}{\partial t} u(t,s)\Big|_{t=0} = \sum_{j=1}^{\infty} x_{2j}(0)\phi_j(s) \tag{3.129}$$

where the set of $\{x_{2j-1}\}$ and $\{x_{2j}\}$ are real value random variables and $\{\phi_j(s)\}$ is an orthonormal set in $L^2(0,1)$ which are to be specified in the sequel.

Then

$$\bar{P}_0(s_1,s_2) = \begin{bmatrix} p_0^{11}(s_1,s_2) & p_0^{12}(s_1,s_2) \\ p_0^{21}(s_1,s_2) & p_0^{22}(s_1,s_2) \end{bmatrix}$$

$$= E\left\{ \begin{bmatrix} \sum_{i=1}^{\infty} x_{2i-1}(0)\phi_i(s_1) \\ \sum_{i=1}^{\infty} x_{2i}(0)\phi_i(s_1) \end{bmatrix} \begin{bmatrix} \sum_{k=1}^{\infty} x_{2k-1}(0)\phi_k(s_2) & \sum_{k=1}^{\infty} x_{2k}(0)\phi_k(s_2) \end{bmatrix} \right\} \tag{3.130}$$

By assuming that the set of random variables $\{x_{2j-1}\}$ and $\{x_{2j}\}$ are orthonormal it follows that

$$\bar{P}_0(s_1,s_2) = \sum_{i=1}^{\infty} \begin{bmatrix} x_{2i-1}^2(0) & x_{2i-1}(0)x_{2i}(0) \end{bmatrix} \phi_i(s_1)\phi_i(s_2)$$

$$\lfloor x_{2i}(0)x_{2i-1}(0) \quad x_{2i}^2(0) \rfloor$$

$$= \sum_{i=1}^{\infty} P_i(0)p_i(s_1,s_2) \tag{3.131}$$

Substituting these relationships into the first three terms of (3.127), one has

$$\sum_{i=1}^{\infty} \sum_{j=1}^{n} \int_0^1 ds_1' \int_0^1 ds_2' \phi_j(s_1')\phi_j(s_2')p_i(s_1,s_2)$$
$$E\left\{[x_{2i-1}(0)\cos\omega_j ct + \frac{x_{2i}(0)}{\omega_j c} \sin\omega_j ct]^2\right\} \tag{3.132}$$

which needs to be maximized for every fixed t 0. From the result given in Sec. 2.3, it is apparent that the optimum set of basis function $\{\phi_j(s)\}$ is the eigenfunction of the homogeneous integral equation,

$$\begin{bmatrix} p_{2j-1,2j-1}(0) & p_{2j-1,2j}(0) \\ p_{2j,2j-1}(0) & p_{2j,2j}(0) \end{bmatrix} \phi_j(s) = P_j(0) \int_0^1 p_j(s,s')\phi_j(s')ds' \tag{3.133}$$

where P.,.(0) is the corresponding eigenvalues which represent the initial amount of uncertainty associated with each coordinate function $\phi_j(s)$. From (2.47) the covariance kernel $\bar{P}_0(s,s')$ may be expressed uniformly:

$$\bar{P}_0(s,s') = \sum_{j=1}^{\infty} \begin{bmatrix} p_{2j-1.2j-1}(0) & p_{2j-1,2j}(0) \\ p_{2j,2j-1}(0) & p_{2j,2j}(0) \end{bmatrix} \phi_j(s)\ \phi_j(s') \tag{3.134}$$

By using this result in (3.128) it follows that

$$P^*(t) = \sum_{j}^{n} \Big[p_{2j-1,2j-1}(0)\cos^2\omega_j ct + p_{2j,2j}(0)\frac{1}{\omega_j^2 c^2}\sin^2\omega_j ct$$

$$(p_{2j-1,2j}(0) + p_{2j,2j-1}(0)\ \frac{1}{\omega_j c}\ \cos\omega_j ct\ \sin\omega_j ct$$

$$+ \frac{1}{\omega_j^2 c^2} \int_0^t dt' \sin^2 \omega_j c(t-t')[c^4(q_{00}(t')\phi_j(0) + q_{01}(t')\phi_j(0)\phi_j(1)$$

$$+ q_{10}(t')\phi_j(0)\phi_j(1) + q_{11}(t')\phi_j^2(1))] + Q(t')\Big] \tag{3.135}$$

where p*(t) is defined to be the variance of the finite dimensional modal approximation of the state u(t,s).

The modal representation of the random initial-boundary-value wave problem is found from the relationship given by (3.56),

$$x_{2j-1}(t) = \langle u(t,s),\ \phi_j(s) \rangle \qquad 0 \le s \le 1\ ;\ j = 1,2,\ldots,n \tag{3.136}$$

Substituting the Green's function solution form of u(t,s), (2.31),into (3.136) gives

$$x_{2j-1}(t) = \int_0^1 ds\phi_j(s)\Bigg[\int_0^1 \left(\sum_{i=1}^{\infty} \frac{\sin\omega_i ct}{\omega_i c} \phi_i(s)\phi_i(s')\right) u_1(s')ds'$$

$$+ \int_0^1 \left(\sum_{i=1}^{\infty} \cos\omega_i ct\ \phi_i(s)\phi_i(s')\right) u_0(s')ds'$$

$$+ c^2 \int_0^t \sum_{i=1}^{\infty} \frac{\sin\omega_i c(t-t')}{\omega_i c} \phi_i(s)\Big(\phi_i(0)v_0(t') + \phi_i(1)v_1(t')\Big)dt'$$

$$+ \int_0^t dt' \int_0^1 ds'\ w(t',s') \sum_{i=1}^{\infty} \frac{\sin\omega_i c(t-t')}{\omega_i c} \phi_i(s)\phi_i(s')\Bigg]$$

$$\tag{3.137}$$

where the series expansion of Green's function (2.32) and the properties indicated by (3.17) and (3.18) are utilized. By performing the integration over $0 \le s \le 1$ and using the definition of orthonormal functions, (3.137) becomes

$$x_{2j-1}(t) = x_{2j}(0)\sin\omega_j ct + x_{2j-1}(0)\cos\omega_j ct$$
$$+ \frac{1}{\omega_j c}\int_0^1 \sin\omega_j c(t-t')[c^2\phi_j(0)v_0(t')$$
$$+ c^2\phi_j(1)v_1(t') + w_j(t')]\, dt' \qquad (3.138)$$

where

$$x_{2j-1}(0) = \int_0^1 u(0,s')\,\phi_j(s')\, ds' \qquad (3.139)$$

$$x_{2j}(0) = \frac{1}{\omega_j c}\int_0^1 \frac{\partial}{\partial t}u(t,s')\Big|_{t=0}\phi_j(s')\, ds' \qquad (3.140)$$

and

$$w_j(t) = \int_0^1 w(t',s')\,\phi_j(s')\, ds' \qquad (3.141)$$

By defining

$$x_{2j}(t) = \frac{1}{\omega_j c}\frac{d}{dt}x_{2j-1}(t) \qquad (3.142)$$

it is seen that

$$x_{2j}(t) = x_{2j}(0)\cos\omega_j ct - x_{2j-1}(0)\sin\omega_j ct$$

$$+ \frac{1}{\omega_j c}\int_0^t \cos\omega_j c(t-t')[c^2\phi_j(0)v_0(t') + c^2\phi_j(1)v_1(t') + w_j(t')]\, dt' \tag{3.143}$$

From (3.138) and (3.143)

$$\begin{bmatrix} x_{2j-1}(t) \\ x_{2j}\ \ (t) \end{bmatrix} = \begin{bmatrix} \cos\omega_j ct & \sin\omega_j ct \\ -\sin\omega_j ct & \cos\omega_j ct \end{bmatrix} \begin{bmatrix} x_{2j-1}(0) \\ x_{2j}\ \ (0) \end{bmatrix} \tag{3.144}$$

$$+ \frac{1}{\omega_j c}\int_0^t \begin{bmatrix} \cos\omega_j c(t-t') & \sin\omega_j c(t-t') \\ -\sin\omega_j c(t-t') & \cos\omega_j c(t-t') \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ c^2\phi_j(0) & c^2\phi_j(1) & 1 \end{bmatrix} \begin{bmatrix} v_0(t') \\ v_1(t') \\ w_j(t') \end{bmatrix} dt'$$

This expression is equivalent to

$$x^j(t) = \Phi_j(t)x^j(0) + \int_0^t \Phi_j(t-t')D_j w^j(t')\, dt' \tag{3.145}$$

where $\Phi_j(t)$ is the state transition matrix satisfying

$$\Phi_j(t-t_0) = A_j\Phi_j(t-t_0) \tag{3.146}$$

$$\Phi_j(t_0,t_0) = I \tag{3.147}$$

for a linear system

$$\dot{x}^j(t) = A_j x^j(t) + D_j w^j(t) \tag{3.148}$$

$$x^j(0) = \begin{bmatrix} x_{2j-1}(0) \\ x_{2j}\ \ (0) \end{bmatrix} \tag{3.149}$$

The following relationships exist between expressions (3.144) and (3.145) and (3.149):

$$\Phi_j(t,t_0) = \begin{bmatrix} \cos\omega_j c(t-t_0) & \sin\omega_j c(t-t_0) \\ -\sin\omega_j c(t-t_0) & \cos\omega_j c(t-t_0) \end{bmatrix} \tag{3.150}$$

$$A_j = \begin{bmatrix} 0 & \omega_j c \\ -\omega_j c & 0 \end{bmatrix} \qquad D_j = \begin{bmatrix} 0 & 0 & 0 \\ c^2\phi_j(0) & c^2\phi_j(1) & 1 \end{bmatrix} \tag{3.151}$$

$$x^j(t) = \begin{bmatrix} x_{2j-1}(t) \\ x_{2j}\ \ (t) \end{bmatrix} \qquad w^j(t) = \begin{bmatrix} v_0(t) \\ v_1(t) \\ w_j(t) \end{bmatrix} \tag{3.152}$$

where $t \geq 0$ and $j=1,2,---n$. Since $x^j(0)$ and $w^j(t)$ are random quantities the statistical properties need to be characterized. The expectations are

$$E\{x^j(0)\} = E\left\{ \begin{bmatrix} u(0,s') & \phi_j(s') \\ \frac{\partial}{\partial t}u(t,s')\Big|_{t=0} , & \phi_j(s') \end{bmatrix} \right\} = 0 \tag{3.153}$$

$$E\{w^j(t)\} = 0 \tag{3.154}$$

the variance terms for the initial conditions are

$$E\left\{x^j(0)x^{k^T}(0)\right\} = E\left\{ \begin{bmatrix} <u(0,s) , & \phi_j(s)> \\ <\frac{\partial}{\partial t} u(t,s)\Big|_{t=0} , & \phi_j(s)> \end{bmatrix} \begin{bmatrix} <u(0,s') , \phi_k(s')> <\frac{\partial}{\partial t}u(t,s')\Big|_{t=0} , \phi_k(s') > \end{bmatrix} \right\}$$

$$= \begin{bmatrix} p_{2j-1,2j-1}(0) & p_{2j-1,2j}(0) \\ p_{2j,2j-1}(0) & p_{2j,2j}(0) \end{bmatrix} \delta_{jk} = P_j(0)\ \delta_{jk} \tag{3.155}$$

and the covariance for the random disturbances is

$$E\left\{\begin{bmatrix} v_0(t) \\ v_1(t) \\ w_j(t) \end{bmatrix} \begin{bmatrix} v_0(t') & v_1(t') & w_k(t') \end{bmatrix}\right\}$$

$$= \begin{bmatrix} q_{00}(t) & q_{01}(t) & 0 \\ q_{10}(t) & q_{11}(t) & 0 \\ 0 & 0 & Q(t)\delta_{jk} \end{bmatrix} \delta(t-t') = Q_j(t)\,\delta(t-t') \tag{3.156}$$

The 2n dimensional vector expression of the system becomes

$$\dot{x}(t) = Ax(t) + D\bar{w}(t) \tag{3.157}$$

$$x(0) = x_0 \tag{3.158}$$

$$E\{x(0)\} = 0 \qquad E\{\bar{w}(t)\} = 0 \tag{3.159}$$

$$E\{x(0)x^T(0)\} = \text{diag}\,[P_1(0)\ P_2(0)\ \ldots\ P_n(0)] \tag{3.160}$$

$$E\{\bar{w}(t)\bar{w}^T(t')\} = \begin{bmatrix} q_{00}(t) & q_{01}(t) & 0 & \cdots & 0 \\ q_{10}(t) & q_{11}(t) & 0 & \cdots & 0 \\ 0 & 0 & Q(t) & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Q(t) \end{bmatrix} \delta(t-t') \tag{3.161}$$

where

$$A = \text{diag}\,[A_1\ \ A_2\ \ldots\ A_n] \tag{3.162}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ c^2\phi_1(0) & c^2\phi_1(1) & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{3.163}$$

$$\begin{bmatrix} c^2\phi_n(0) & c^2\phi_n(1) & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$x(t) = [x^1(t) \quad x^2(t) \quad \dots \quad x^n(t)\ ]^T \tag{3.164}$$

and

$$\bar{w}(t) = [v_0(t) \quad v_1(t) \quad w_1(t) \ \dots \ w_n(t)]^T \tag{3.165}$$

Therefore, the random field solution of the stochastic initial-boundary-value wave problem defined in Sec. 3.2.2 may be expressed in terms of the finite mode expression:

$$u^*(t,s) = \sum_{j=1}^{n} x_{2j-1}(t)\phi_j(s) \tag{3.166}$$

where $u^*(t,s)$ converges to $u(t,s)$ in the mean and $x_{2j-1}(t)$ is defined as the solution of (3.157) to (3.165). As discussed in the previous section the set of optimal basis functions $\{\phi_j(s)\}$ is obtained by solving the set of homogeneous integral equations (3.133). The resulting basis functions also satisfy the Sturm-Liouville equation with homogeneous boundary conditions, (3.98) to (3.100), when the initial-boundary problem is well-posed.

The above analysis may be easily extended to the random initial-value problem with periodic stochastic processes in space, $-\infty<s<\infty$, at t=0, i.e.,

$$\bar{P}_0(\sigma) = \bar{P}_0(\sigma+1) \tag{3.167}$$

where $\sigma=s-s'$ and the period is one. The result constitutes a special case of the outcome for the initial-boundary-value problem, in which there is no boundary noise term. In studying the spatial mode representation for a randpm periodic precess of time in the folling section, an analogous procedure will be presented.

3.5.2 Spatial Mode Representation

In the preceding two sections the temporal mode representation for initial-value heat or wave problems has been discussed. If a change of independent parameter is made from s to t, the treatment of the spatial mode representation in this section can equally serve for the study of temporal mode representation of random initial-value problems. Suppose the solution u(t,s) of a random one-point-boundary value problem, (3.43) to (3.46), is knoun to demonstrate a periodic random process in time and the Markov property in the spatial region of a semi-infinite interval, $s \geq 0$. Then, in contrast to the case in the preceding random-initial-boundary problems, the dependence upon time may be prescribed by a set of n-dimensional orthonormal functions $\{\phi_j(t)\}$ and the dependence upon space is restricted to an explicit dependence upon n spatial stochastic processes, $\{x_{2j-1}(s)\}$, i.e.,

$$u^*(t,s) = \sum_{j=1}^{n} x_{2j-1}(s)\phi_j(t) \qquad -\infty < t < \infty \; ; \quad s \geq 0 \tag{3.168}$$

where the optimum set of orthonormal basis functions $\{\phi_j(t)\}$ is to be found by minimizing the mean squared norm of the error

$$E\{\|u(t,s) - u^*(t,s)\|^2\} = E\{\langle u(t,s), u(t,s)\rangle\}$$

$$- \sum_{j=1}^{n} \int_0^T dt_1 \int_0^T dt_2 E\{u(t_1,s)u(t_2,s)\}\phi_j(t_1)\phi_j(t_2) \tag{3.169}$$

for every fixed $s \geq 0$. The set of random processes $\{x_{2j-1}(s)\}$ is obtained by using

$$x_{2j-1}(s) = \langle u(t,s), \phi_j(t)\rangle \tag{3.170}$$

The norm, $\|\cdot\|$, and the inner product, $<\cdot,\cdot>$ for this case, are defined in the temporal interval [0,T] where T is the time period of the periodic autocovariance of the state at the boundary point s=0,

$$\bar{P}_b(t_1,t_2) = \bar{P}_b(t_1,t_2) = \bar{P}_b(t_1-t_2 + T) \qquad (3.171)$$

To minimize the right hand side of (3.169) with respect to $\{\phi_j(t)\}$ for every fixed s, the second term is maximized. Before substituting the convariance expression

$$E\{u(t_1,s)u(t_2,s)\} = \bar{p}(t_1,s,t_2,s) \qquad (3.172)$$

of (3.48) into the second term of (3.169), one can simplify the expresion. By knowing that $\bar{P}_b(t_1-t_2)$ is periodic it is possible to write a Fourier series expansion of $\bar{P}_b(t_1-t_2)$, i.e.,

$$\bar{P}_b(t_1-t_2) = \sum_{j=-\infty}^{\infty} P_j(0)\, e^{i\omega_j(t_1-t_2)} \qquad (3.173)$$

where

$$P_j(0) = \begin{bmatrix} p_{2j-1,2j-1}(0) & p_{2j-1,2j}(0) \\ p_{2j,2j-1}(0) & p_{2j,2j}(0) \end{bmatrix} = E\left\{\begin{bmatrix} x^2_{2j-1}(0) & x_{2j-1}(0)x_{2j}(0) \\ x_{2j}(0)x_{2j-1}(0) & x^2_{2j}(0) \end{bmatrix}\right\} \qquad (3.174)$$

and $w = 2\pi j/T$.

Then, from (3.48)

$$\bar{p}_{11}(t_1,t_2,s) = \frac{1}{4}\sum_{j=-\infty}^{\infty}\Big[p_{2j-1,2j-1}(0)\, e^{i\omega_j(t_1-t_2)}\big(2 + e^{i\omega_j\frac{2s}{c}} + e^{-i\omega_j\frac{2s}{c}}\big)$$

$$+ c\int_{t_2-s/c}^{t_2+s/c} dt_2'\; p_{2j-1,2j}(0)\big(e^{i\omega_j(t_1+s/c-t_2')} + e^{i\omega_j(t_1-s/c-t_2')}\big)$$

$$+ c\int_{t_2-s/c}^{t_1+s/c} dt' \, p_{2j,2j-1}(0)\left(e^{i\omega_j(t_1'-t_2-s/c)} + e^{i\omega_j(t_1'-t_2+s/c)}\right)$$

$$+ c^2 \int_{t_1-s/c}^{t_1+s/c} dt_1' \int_{t_2-s/c}^{t_2+s/c} dt_2' \, p_{2j,2j}(0) e^{i\omega_j(t_1'-t_2')}\Bigg] \frac{1}{4c^2}\int_0^s ds' \int_{t_1-\frac{s-s'}{c}}^{t_1+\frac{s-s'}{c}} dt_1' \int_{t_2+\frac{s-s'}{c}}^{t_2+\frac{s-s'}{c}} dt_2' \, Q(t_1',t_2',s') \tag{3.175}$$

Integrating the second, third, and fourth term on the right gives

$$\frac{1}{4}\sum_{j=-\infty}^{\infty} e^{i\omega_j(t_1-t_2)}\Big[p_{2j-1,2j-1}(0)\left(e^{i\omega_j s/c} + e^{-i\omega_j s/c}\right)^2$$

$$+ \frac{2c}{i\omega_j} p_{2j-1,2j}(0)\left(e^{i\omega_j s/c} - e^{-i\omega_j s/c}\right)^2 - \frac{c^2}{\omega_j^2} p_{2j-1,2j}(0)\left(e^{i\omega_j s/c} - e^{i\omega_j s/c}\right)^2$$

$$= \frac{1}{4}\sum_{j=-\infty}^{\infty} e^{i\omega_j(t_1-t_2)} E\Big\{\Big[x_{2j-1}(0)\left(e^{i\omega_j s/c} + e^{-i\omega_j s/c}\right)$$

$$- \frac{c}{i\omega_j} x_{2j}(0)\left(e^{i\omega_j s/c} - e^{-i\omega_j s/c}\right)\Big]^2\Big\} \tag{3.176}$$

By rewriting (3.173) in trigonometric form and assuming that (3.17) still holds for

$$\bar{P}_b(t_1-t_2) = \sum_{j=1}^{\infty} P_j(0)\cos\omega_j(t_1-t_2) \tag{3.177}$$

(3.175) becomes

$$\bar{p}_{11}(t_1,s,t_2,s) = \sum_{j=1}^{\infty} \cos\omega_j(t_1-t_2)E\{[x_{2j-1}(0)\cos\omega_j\frac{s}{c} + x_{2j}(0)\sin\omega_j\frac{s}{c}]^2\}$$

$$+ \frac{1}{4c^2}\int_0^s ds' \int_{t_1-\frac{s-s'}{c}}^{t_1+\frac{s-s'}{c}} dt_1' \int_{t_2-\frac{s-s'}{c}}^{t_2+\frac{s-s'}{c}} Q(t_1',t_2',s')\, dt_2' \qquad (3.178)$$

Substituting this into (3.169) and knowing that the term involving the white random field does not affect the error criterion, one obtains the resulting integral equation

$$p_j(s)\,\phi_j(t) = {}_{k=1}\int_0^T p_k \cos\omega_j(t-t')\,\phi_j(t')\,dt'$$

$$0 \le t \le T \quad ; j = 1, 2, \ldots n \qquad (3.179)$$

which maximizes the quantity

$$\sum_{j=1}^{n}\sum_{k=1}^{\infty} E\left\{[x_{2k-1}(0)\cos\omega_k\frac{s}{c} + x_{2k}(0)\sin\omega_k\frac{s}{c}]^2\right\}\int_0^T dt_1\int_0^T dt_2 \cos\omega_k(t_1-t_2)\phi_j(t_1)$$

$$\cdot\phi_j(t_2) \qquad (3.180)$$

where

$$p_k \triangleq E\left\{[x_{2k-1}(0)\cos\phi_k s/c + x_{2k}(0)\sin\phi_k s/c]^2\right\} \qquad (3.181)$$

is defined for every fixed s and the term by term maximization for each j is obtained. By solving (3.179) it follows that the eigenfunctions are given by

$$\phi_j(t) = \sqrt{\frac{2}{T}}(\cos\omega_j t + \sin\omega_j t) \qquad j = 1, 2, \ldots n \qquad (3.182)$$

with the eigenvalues

$$p_j(0) = p_j \qquad j = 1, 2, \ldots n \tag{3.183}$$

Thus, the optimum set of basis functions decomposing the random time periodic field is found to be an orthonormal set of harmonic functions. The term $p_j(0)$ denotes the energy content of each mode at each spatial location s. Using (3.177) and (3.182) in the second term on the right of (3.169) one obtains the expression for the variance of the finite dimensional modal approximation of the state u(t,s).

$$p^*(s) = \sum_{j=1}^{n} E\left\{[x_{2j-1}(0)\cos\omega_j \frac{s}{c} + x_{2j}(0)\sin\omega_j \frac{s}{c}]^2\right\} + \frac{1}{\omega_j^2 c^2}\int_0^s \sin^2\omega_j \frac{s-s'}{c} Q(s')ds' \tag{3.184}$$

The spatial mode representation of the one point boundary-value problem is obtained by substituting the Green's function solution form of u(t,s) into (3.170):

$$x_{2j-1}(s) = \left\langle \frac{1}{2}[u(t+s/c, 0) + u(t-s/c, 0)] + \frac{c}{2}\int_{t-s/c}^{t+s/c} \frac{\partial}{\partial s} u(t,s)\Big|_{s=0} dt' \right.$$

$$\left. + \frac{1}{2c}\int_0^s ds' \int_{t-\frac{s-s'}{c}}^{t+\frac{s-s'}{c}} w(t',s')\, dt' \,,\ \phi_j(t) \right\rangle \tag{3.185}$$

$$j = 1, 2, \ldots n$$

By using

$$u(t,s) = \sum_{k=1}^{\infty} x_{2k-1}(s)\phi_k(t) = \sum_{k=1}^{\infty} x_{2k-1}(s)\sqrt{2/T}(\cos\omega_k t + \sin\omega_k t) \tag{3.186}$$

and

$$w(t,s) = \sum_{k=1}^{\infty} w_k(s)\phi_k(t) = \sum_{k=1}^{\infty} w_k(s)\sqrt{2/T}(\cos\omega_k t + \sin\omega_k t) \tag{3.187}$$

in (3.185), it follows that

$$x_{2j-1}(s) = x_{2j-1}(0)\cos\omega_j\frac{s}{c} + x_{2j}(0)\sin\omega_j\frac{s}{c} + \frac{1}{\omega_j c}\int_0^s \sin\omega_j\frac{s-s'}{c}\, w_j(s')\, ds' \tag{3.188}$$

where

$$x_{2j-1}(0) = \int_0^T u(t',0)\phi_j(t')\, dt' \tag{3.189}$$

$$x_{2j}(0) = \frac{c}{\omega_j}\int_0^T \frac{\partial}{\partial s} u(t',s)\Big|_{s=0} \phi_j(t')\, dt' \tag{3.190}$$

and

$$w_j(s) = \int_0^T w(t',s')\phi_j(t')\, dt' \tag{3.191}$$

By defining

$$x_{2j}(s) = \frac{c}{\omega_j}\frac{d}{ds} x_{2j-1}(s) \tag{3.192}$$

one has

$$\begin{bmatrix} x_{2j-1}(s) \\ x_{2j}\ \ (s) \end{bmatrix} = \begin{bmatrix} \cos\omega_j\frac{s}{c} & \sin\omega_j\frac{s}{c} \\ -\sin\omega_j\frac{s}{c} & \cos\omega_j\frac{s}{c} \end{bmatrix} \begin{bmatrix} x_{2j-1}(0) \\ x_{2j}\ \ (0) \end{bmatrix}$$

$$+ \frac{c}{\omega_j}\int_0^s \begin{bmatrix} \cos\omega_j\frac{s-s'}{c} & \sin\omega_j\frac{s-s'}{c} \\ -\sin\omega\frac{s-s'}{c} & \cos\omega\frac{s-s'}{c} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{c^2} \end{bmatrix} w_j(s')\, ds' \tag{3.193}$$

Following the steps analogous to those given in (3.145) to (3.150), with the exchange of parameters t and s one obtains the 2n dimensioned vector expression of the spatial mode representation

$$\frac{d}{ds}\, x(s) = Ax(s) + D\bar{w}(s) \tag{3.194}$$

$$x(0) = x_0 \tag{3.195}$$

$$E\{x(0)\} = 0 \quad E\{\bar{w}(s)\} = 0 \tag{3.196}$$

$$E\{x(0)x^T(0)\} = \text{diag}\ [\ P_1(0),\ P_2(0),\ \ldots\ P_n(0)\ ] \tag{3.197}$$

$$E\{\bar{w}(s)\bar{w}^T(s')\} = \text{diag}\ [\ Q_1(s),\ Q_2(s),\ \ldots\ Q_n(s)\ ]\delta(s-s') \tag{3.198}$$

where

$$A = \text{diag}\begin{bmatrix} 0 & \frac{\omega_1}{c} & 0 & \frac{\omega_2}{c} & & 0 & \frac{\omega_n}{c} \\ -\frac{\omega_1}{c} & 0 & -\frac{\omega_2}{c} & 0 & \cdots & -\frac{\omega_n}{c} & 0 \end{bmatrix} \tag{3.199}$$

$$D = [\ 0,\ \frac{1}{c^2},\ 0,\ \frac{1}{c^2},\ \ldots\ 0,\ \frac{1}{c^2}\ ]^T \tag{3.200}$$

$$x(s) = [x_1,\ x_2,\ x_3,\ x_4,\ \ldots\ x_{2n-1},\ x_{2n}]^T \tag{3.201}$$

and

$$\bar{w}(s) = [w_1(s),\ w_2(s),\ \ldots\ w_n(s)]^T \tag{3.202}$$

Thus, the solution of the random one point boundary-value wave problem given by (3.43) to (3.46) may be represented in random finite Fourier series expansion,

$$u^*(t,s) = \sum_{j=1}^{n} x_{2j-1}(s)\phi_j(t) \tag{3.203}$$

where $u^*(t,s)$ converges to $u(t,s)$ in the mean as $n \to \infty$ and the set of spatial Markov process $x_{2j-1}(s)$ is the solution of the system of random ordinary differential equation defined by (3.194) to (3.202). This may be called a random inhomogeneous vector Helmholtz's equation. It is noted that the temporal representation of the random initial-value wave problem follows the same procedure as that taken in this section.

3.5.3 Illustrative Example

For clarity of exposition, the following example is studied. Consider the radiation of scalar waves by a random periodic point source in a lossless, homogeneous isotropic time-independent medium. The random field $u(t,s)$ represents the amplitude of the wave at any point (t,s), having an expected functional form $\mu_0(t)$ and an expected functional form of its spatial derivative $\mu_1(t)$ at $s=0$. Random distribution of discrete scatterers and imperfect modeling may produce random disturbances in the system. Let the system model be given by:

$$\frac{\partial^2}{\partial s^2} u(t,s) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(t,s) + w(t,s) \qquad -\infty < t < \infty \; ; \quad s \geqslant 0 \tag{3.204}$$

where c is the refractive index and $w(t,s)$ is a white Gaussian random field with the statistical properties given by

$$E\{w(t,s)\} = 0 \tag{3.205}$$

$$E\{w(t,s)w(t',s')\} = Q\,\delta(t-t',s-s')$$
$$-\infty < t < \infty \quad ; \quad s \geq 0 \tag{3.206}$$

The one-point boundary conditions are Gaussian with the statistics given by

$$E\{u(t,0)\} = 0 \tag{3.207}$$

$$E\{u_s(t,0)\} = 0 \tag{3.208}$$

$$E\left\{\begin{bmatrix} u(t,0) \\ u_s(t,0) \end{bmatrix} [u(t',0) \quad u_s(t',0)]\right\} = \bar{P}_b(t,t')$$
$$= \sum_{j=1}^{\infty} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{\omega_j^2} \cos\omega_j(t-t') \tag{3.209}$$

where $u_s(t,0) = \frac{\partial}{\partial s}u(t,s)\Big|_{s=0}$.

To find the modal representation of the problem the covariance $\bar{P}_b(t,t')$ at the boundary, s=0, is taken into the homogeneous equation (3.179) and the equation is solved for its eigenvalues and eigenfunctions, i.e., the following equation must be satisfied

$$P_j(0)\phi_j(t) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \int_0^T p' \sum_{i=1}^{\infty} \frac{1}{\omega_i^2} \cos\omega_i(t-t')\phi_j(t')\, dt'$$
$$j = 1, 2, \ldots \tag{3.210}$$

where the set of eigenfunctions can be obtained to be $\{\sqrt{2/T}(\cos\omega_j t + \sin\omega_j t)\}$ with the set of eigenvalues $\{p'/\omega_j^2\}$ for each $j = 1, 2, \ldots$.

The solution of the stochastic wave problem can then be approximated by taking the first two terms as

$$u(t,s) = \sum_{j=1}^{2} x_{2j-1}(s)\sqrt{2/T}(\cos\omega_j t + \sin\omega_j t) \tag{3.211}$$

where the state $x_1(s)$ and $x_3(s)$ are the solutions of the following stochastic ordinary differential system:

$$\frac{d}{ds}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\omega_1}{c} & 0 & 0 \\ -\frac{\omega_1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\omega_2}{c} \\ 0 & 0 & -\frac{\omega_2}{c} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{3.212}$$

which is charaterized by the statistical properties:

$$E\{x(0)\} = 0 \tag{3.213}$$

$$E\{x(0)x^T(0)\} = p'\begin{bmatrix} 1/\omega_1^2 & 1/\omega_1^2 & 0 & 0 \\ 1/\omega_1^2 & 1/\omega_1^2 & 0 & 0 \\ 0 & 0 & 1/\omega_2^2 & 1/\omega_2^2 \\ 0 & 0 & 1/\omega_2^2 & 1/\omega_2^2 \end{bmatrix} \tag{3.214}$$

$$E\{w_1(s)\} = E\{w_2(s)\} = 0 \tag{3.215}$$

$$E\begin{bmatrix} w_1(s) \\ w_2(s) \end{bmatrix}\begin{bmatrix} w_1(s) & w_2(s') \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}\delta(s-s') \tag{3.216}$$

Note that the resulting modal representation yields the statistically independent set of ordinary random differential equations. The state estimation problem for this example will be studied in Chapter IV.

3.6 Error Bound Estimates

The mean square convergence, convergence in quadratic mean, is known to be ensured, provided the set of basis functions constitutes a complete orthonormal sequence. To study the error estimates of the finite approximation, the performance measure used in approximating the random field is to be examined:

For a random heat problem, by substituting (3.76) into (3.64) where $t=t'$ it follows that

$$E\{||u(t,s) - u^*(t,s)||^2\} = \sum_{j=n+1}^{\infty} p_j(t)$$

$$= \sum_{j=n+1}^{\infty} \left[p_j(0)e^{-2k\lambda_j t} + \int_0^t dt'\ e^{-2k\lambda_j(t-t')} [k^2(\phi_j^2(0)q_{00}(t') \right. \tag{3.217}$$

$$\left. + \phi_j(0)\phi_j(1)(q_{01}(t') + q_{10}(t')) + \phi_j^2(1)q_{11}(t') + Q(t')] \right]$$

Since $q_{00}(t)$, $q_{01}(t)$, $q_{10}(t)$, $q_{11}(t)$, and $Q(t)$ are assumed to be bounded real function on $t \geq 0$, there exist a real number,

$$M = \text{lub}\left\{k^2[\phi_j^2(0)q_{00}(t) + \phi_j(0)\phi_j(1)(q_{01}(t)+q_{10}(t)) + \phi_j^2(1)q_{11}(t) + Q(t)]\right\} \tag{3.218}$$

where lub is a least upper bound for every $t \geq 0$ and $j=n+1$, $n+2$,---.

Then, from (3.217)

$$p'(t) \leq M\int_0^t e^{-2k\lambda_j(t-t')} dt' = \frac{M}{2k\lambda_j}[1 - e^{-2k\lambda_j t}]$$

$$\leq \frac{M}{2k\lambda_j} \tag{3.219}$$

where $p_j'(t)$ denotes the amount of uncertainty contributed by random disturbance. Thus

$$\{||u(t,s) - u^*(t,s)||^2\} = \sum_{j=n+1}^{\infty} p_j(t) \leq \sum_{j=n+1}^{\infty} (p_j(0) + \frac{M}{2k\lambda_j}) \quad (3.220)$$

a closed form representation (3.220) is possible because $p_j(0) + M/2k\lambda_j$ is a monotone decreasing series. A well known formula for monotone decreasing series states that [28],

$$\sum_{j=n+1}^{\infty} \alpha_j \leq \int_n^{\infty} f(x')\, dx' \qquad (3.221)$$

where α_j is a monotone decresaing sequence and $f(x')$ is a function such that

$$\alpha_j = f(j) \qquad j = 1, 2, \ldots \qquad (3.222)$$

For the example problem illustrated in Section 3.4 it was found that

$$\phi_j(s) = \sqrt{2} \sin(j + 1/2)\pi s \qquad (3.223)$$

$$p_j(0) = \bar{p}_0/(j + 1/2)^2\pi^2 \qquad (3.224)$$

and

$$\lambda_j = (j + 1/2)^2\pi^2 \qquad (3.225)$$

Therefore

$$E\{ ||u(t,s) - u^*(t,s)||^2\} \leq \sum_{j=n+1}^{\infty} (p_j(0) + \frac{M}{2k\lambda_j})$$

$$\leq \int_n^{\infty} \frac{2k\bar{p}_0 + M}{2k(\frac{2x+1}{2})^2\pi^2}\, dx = (\frac{2k\bar{p}_0 + M}{k\pi^2})\ \frac{1}{2n+1} \qquad (3.226)$$

which implies that the error energy decreases on the order of $1/(2n+1)$.

For the wave problem the stochastic one point boundary-value wave problem is to be considered. From (3.123) and (3.184)

$$E\{\|u(t,s) - u^*(t,s)\|^2\} = \sum_{j=n+1}^{\infty} p_{2j-1,2j-1}(s)$$

$$= \sum_{j=n+1}^{\infty} \Big[E\Big\{[x_{2j-1}(0)\cos\omega_j \frac{s}{c} + x_{2j}(0)\sin\omega_j \frac{s}{c}]^2\Big\} + \frac{1}{\omega_j^2 c^2}\int_0^s \sin\omega_j \frac{s-s'}{c} Q(s')\, ds' \Big] \qquad (3.227)$$

where $p_{2j-1,2j-1}(s)$ is an element of the matrix $P_j(s)$. Since $Q(s)$ is a bounded real function on $s \geq 0$ there exists

$$M = \text{lub } Q(s) \qquad (3.228)$$

for every $s \geq 0$. Then from (3.227) and knowing the fact that periodic signal is a power signal [32], one obtains

$$p_{2j-1,2j-1}(s) \leq E\,[x_{2j-1}(0) + x_{2j}(0)]^2 + \frac{M}{\omega_j c S_j}\int_0^{S_j} \sin\omega_j \frac{s-s'}{c}\, ds'$$

where S_j denotes the period of the j-th signal, i.e., $S_j = 2\pi c/\omega_j$. By performing the integration on the right and substituting into (3.227)

$$E\left\{\|u(t,s) - u^*(t,s)\|^2\right\} \tag{3.229}$$

$$\leq \sum_{j=n+1}^{\infty} \left[p_{2j-1,2j-1}(0) + 2p_{2j-1,2j}(0) + p_{2j,2j}(0) + \frac{M}{2\omega_j^2 c^2}\right]$$

Since $P_j(0)$ is given for an initial condition, and for most practical purposes it is assumed to be a monotone decreasing sequence, equation (3.229) may be expressed as

$$E\left\{\|u(t,s) - u^*(t,s)\|^2\right\} \leq \int_n^{\infty} f(x')\,dx' \tag{3.230}$$

If

$$\alpha_j = p_{2j-1,2j-1}(0) + 2p_{2j-1,2j}(0) + p_{2j,2j}(0) + \frac{M}{2\omega_j^2 c^2} \tag{3.231}$$

is a monotone decreasing sequence, then f(x') in (3.230) is a function such that

$$f(j) = \alpha_j \qquad j = n+1,\ n+2,\ .\ .\ . \tag{3.232}$$

For example, if $\bar{P}_b(t,t')$ is given as

$$\bar{P}_b(t-t') = \sum_{j=1}^{\infty} \frac{1}{\omega_j^2} \cos\omega_j(t-t') \tag{3.233}$$

where $\omega_j=2\pi j/T$, the finite n-mode approximation minimizing the mean squared norm of error, $E\{\|u(t,s)-u^*(t,s)\|^2\}$, yields the bound of error given by

$$E\left\{\|u(t,s) - u^*(t,s)\|^2\right\} \leq \int_n^{\infty} T^2\left[\frac{4}{(2\pi x')^2} + \frac{M}{2c^2(2\pi x')^2}\right] dx'$$

$$\leq \frac{T^2(8c^2 + M)}{8c^2\pi^2 n} \tag{3.234}$$

which implies that the power of the error decreases on the order of $1/n$.

From a practical point of view, knowledge of the error bound is not a necessary condition for a method to be useful in the solution of engineering problems. In actual applications, the fact that supplementary terms in the truncated functional series $u^*(t,s)$ have a variance decreasing as n increases can be interpreted as being a sufficient condition for the utility of the modal representation. The result obtained in this section will be useful on Chapter IV, yielding an expression which shows the overall performance of a finite mode approximation and stochastic estimation scheme done simultaneously.

3.7 Summary

The representation of stochastic distributed parameter systems has been presented with enough generality to encompass initial-value and initial-boundary-value problems for random diffusion or heat processes and random wave processes. One-point-boundary-value problems were considered for the random wave processes. The mean and covariance of the random solution field, $u(t,s)$ were obtained in terms of the impulse response kernel, Green's function, of each system. On the basis of this result and the minimum mean squared error measure, the modal representation of random distributed parameter systems was developed. It is emphasized that the optimal set of basis functions to be used in decomposing the random system is required to be the set of eigenfunctions for the homogeneous integral equation which possesses the initial covariance or one-point boundary covariance function as the kernel of the equation.

Two illustrative examples for heat and wave process, respectively, were considered and will be utilized in the following chapter. In using $u^*(t,s)$ as an approximation of $u(t,s)$, it is desirable to know how many modes should be considered so that the resulting error of the modal representation does not exceed a certain percentage of the average power of $u(t,s)$. From the mean square convergence property of the approximation, error bounds were derived for the heat and wave problems to provide guidance in the number of modes to be used.

CHAPTER IV

MODAL ESTIMATION

4.1 Introduction

In Chapter III, the modal representations of a class of stochastic distributed parameter systems was developed resulting in a set of random ordinary differential equations.

In this chapter state estimation problems for corresponding systems which evolves in time or in space are to be studied. Since the ability to reconstruct the state using an observation with random measurement noise is related to the stochastic observability of a system, it is assumed that the measurement systems and the sensor locations in this study meet the requirements discussed for n-mode observability [30] and the stochastic observability conditions described in Gelb [33].

Section 4.2 is intended to provide the state reconstruction scheme when the dynamics of the temporal modes are available along with a measurement process. The modal representation presented in Section 3.4 leads to the application of estimation algorithms developed in the area of lumped parameter systems. For the reconstruction of spatial modes, two different signal processing techniques are discussed in Section 4.3. The differences of the algorithms stems from the formulation of the problem and the basic assumptions made on the system.

In Section 4.4 the application of the modal estimation technique to the state reconstruction problems for heat and wave processes is outlined together with the computer simulation results.

4.2 State Estimation for Temporal Dynamics

In the following, linear state estimation algorithms developed for lumped parameter systems, and reviewed in Section 2.4, are applied to generate a filtered or smoothed estimate of the solution of the D.P.S. from a noisy measurement process of a linear form. The measurements are at discrete spatial locations and taken continuously over time.

The systems of stochastic ordinary differential equations derived in Section 3.4 and 3.5.1 describe the temporal evolution of random heat processes and random wave processes, respectively. The spatial dependence of each process is prescribed by the set of optimum orthonormal coordinate functions.

To complete the formulation of state estimation problem it is assumed that noisy measurements are taken by m discrete sensors imbedded in the spatial domain. The observation equation is

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ \cdot \\ \cdot \\ \cdot \\ z_m(t) \end{bmatrix} = \begin{bmatrix} u(t,s_1) \\ u(t,s_2) \\ \cdot \\ \cdot \\ \cdot \\ u(t,s_m) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \\ \cdot \\ \cdot \\ \cdot \\ v_m(t) \end{bmatrix} \tag{4.1}$$

where $\{v_j(t)\}$ is defined to be a white Gaussian noise characterized by

$$E\{v_i(t)\} = 0 \tag{4.2}$$

$$E\{v_i(t)v_k(t')\} = r_{ik}(t)\ \delta(t-t') \tag{4.3}$$

It is important to point out that the actual measurement process described by (4.1) contains the exact state $u(t,s_i)$ of the message process, while one's intention in the modal representation is to deal with a finite number of dominant modes. Fortunately, the advantage of a modal decomposition is that, in most physical systems, higher harmonics of spatial modes usually can be neglected [31], and only a few of the modes have a significant contribution to the behavior of the system [30]. Furthermore, mechanical or electrical sensors for most engineering applications have band-limited characteristics which make the device more sensitive to a certain band of frequency modes and rejects others.

Therefore, it is possible to determine the number of modes, n, to be considered in the finite modal representation, such that the measurement equation (4.1) may be replaced by

$$z(t) = Cx(t) + v(t) \tag{4.4}$$

where x is an n or 2n vector of dominant temporal modes, z is an m vector of observation, v an m vector of white Gaussian noise characterized by

$$E\{v(t)\} = 0 \tag{4.5}$$

$$E\{v(t)v^T(t')\} = R(t)\,\delta(t-t') \tag{4.6}$$

The observation matrix C is an m x n or m x 2n given by

$$C = \begin{bmatrix} \phi_1(s_1) & \phi_2(s_1) & \cdots & \phi_n(s_1) \\ \phi_1(s_2) & \phi_2(s_2) & \cdots & \phi_n(s_2) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \phi_1(s_m) & \phi_2(s_m) & \cdots & \phi_n(s_m) \end{bmatrix} \tag{4.7}$$

for the heat problem or

$$C = \begin{bmatrix} \phi_1(s_1) & 0 & \phi_2(s_1) & 0 & \cdots & \phi_n(s_1) & 0 \\ \phi_1(s_2) & 0 & \phi_2(s_2) & 0 & \cdots & \phi_n(s_2) & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_1(s_m) & 0 & \phi_2(s_m) & 0 & \cdots & \phi_n(s_m) & 0 \end{bmatrix} \tag{4.8}$$

for the wave problem, and $(R(t))_{jk}=r_{jk}(t)$.

In the study of observability for deterministic problems [30], it was seen that the system is n-mode observable if the sensors are not placed at the zeros of any of the basis function $\{\phi_j(s)\}$, yielding an observation matrix C such that none of its first n columns is identically zero for the heat problem. The topic of best sensor locations for a finite-dimensional observer, with respect to minimizing the effect of neglected modes and stochastic measurement errors is discussed in the above reference. It is easy to see that these are the locations that maximizes the signal to noise ratio with respect to the most significant modes and minimizes the signal to noise ratio with respect to the neglected modes.

With the message and observation model defined in the form of a modal representation, the state estimation problem is posed to obtain the filtered estimate of the solution process of a random field $u(t,s)$, $\hat{u}(t,s)$, given a set of basis function $\{\phi_j(s)\}$ and the measurement process $z(t)$ for $t\geq 0$. The performance measure to be used is the $L^2(0,1)\times\Omega$ norm of error, or the mean squared error of the estimate, i.e.,

$$\hat{J} \triangleq E \; ||u(t,s) - \hat{u}(t,s)||^2 \tag{4.9}$$

which is to be minimzed. In Chapter III, the performance measure (4.9) (with u^* instead of $\hat{u}$) was minimzed with respect to every $\phi_j(s)$ in order to choose the optimum set $\{\phi_j(s)\}$ given the characterization of the message process. This has resulted in the expression

$$u(t,s) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{j=1}^{n} x_{j'}(t)\phi_j(s) \tag{4.10}$$

where $j'=j$ for the heat problem and $j'=2j-1$ for the wave problem. The random partial differential equations of $u(t,s)$ are replaced by the set of random ordinary differential equations for the temporal modes $\{x_{j'}(t)\}$. In finding the estimate of $u(t,s)$ given the message process and the measurement process, feasible estimate using the modal representation is described by

$$\hat{u}(t,s) = \sum_{j=1}^{n} x_{j'}(t)\phi_j(s) \tag{4.11}$$

where the optimum choice of the set $\{\phi_j(s)\}$ and the associated coefficients are of interest. Since the set $\{\phi_j(s)\}$ is known from the discussion in Chapter III, substituting (4.10) and (4.11) into the MMSE criterion (4.9) yields

$$\hat{J} = E\left\{ \sum_{j=n+1}^{\infty} x_{j'}^2(t) + \sum_{j=1}^{n} [x_{j'}(t) - \hat{x}_{j'}(t)]^2 \right\} \tag{4.12}$$

which indicates precisely an overall error associated with an estimation process. Interchanging expectation and summation gives

$$\hat{J} = \sum_{j=n+1}^{\infty} E\left\{x_{j'}^2(t)\right\} + \sum_{j=1}^{n} E\left\{[x_{j'}(t) - \hat{x}_{j'}(t)]^2\right\} \tag{4.13}$$

By letting

$$P(t) = E\left\{[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]^T\right\} \qquad (4.14)$$

(4.13) can be rewritten

$$\hat{J} = \sum_{j=n+1}^{\infty} E\left\{x_{j'}^2(t)\right\} + \sum_{j=1}^{\infty} p_{j',j'}(t) \qquad (4.15)$$

where $p_{j',j'}(t)$ is the diagonal element of $P(t)$ for each $j = 1, 2, \ldots n$. Minimization of (4.15) with an estimator of a linear form yields the Kalman filter discussed in Section 2.4.1.

Therefore, the Kalman filtering algorithm and the linear smoothing algorithm presented in Section 2.4 may be applied to obtain a filtered or smoothed estimate of $x(t)$. The equations are rewritten below for convenience:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + K(t)[z(t) - C\hat{x}(t)] \qquad (4.17)$$

$$K(t) = P(t)C^T R(t)^{-1} \qquad (4.18)$$

$$\dot{P}(t) = AP(t) + P(t)A^T + DQ(t)D^T - P(t)C^T R(t)^{-1} CP(t) \qquad (4.19)$$

where the initial conditions are

$$x(0) = E\{x(0)\} = \mu_0 \tag{4.20}$$

$$P(0) = E\left\{[x(0) - \mu(0)][x(0) - \mu(0)]^T\right\} = P_0 \tag{4.21}$$

The state variable form of the modal representation is described by (3.86) through (3.95) for the heat process and by (3.157) through (3.165) for the wave process. The terms involving boundary noise in these models may be deleted for initial-value problems.

The filtered estimate of u*(t,s), is formed as

$$u(t,s) = \sum_{j=1}^{n} x_{j'}(t)\phi_j(s) \tag{4.22}$$

and the mean square error resulting from the filtering operation is

$$E\left\{[u^*(t,s) - u(t,s)]^2\right\} = \phi^T(s)P(t)\phi(s) \tag{4.23}$$

where

$$\phi(s) \triangleq [\phi_1(s) \quad \phi_2(s) \quad . \; . \; . \quad \phi_n(s)]^T \tag{4.24}$$

for heat problems or

$$\phi(s) \triangleq [\,\phi_1(s) \quad 0 \quad \phi_2(s) \quad 0 \quad . \; . \; . \quad \phi_n(s) \quad 0]^T \tag{4.25}$$

for wave problems.

The linear smoothed estimate of x(t) may also be generated using (2.91) through (2.94). Then, the smoothed estimate of the solution to the heat process is

$$u(t,s|T) = \sum_{j=1}^{n} x_{j'}(t|T)\phi_j(s) \tag{4.26}$$

and the smoothed mean square error in estimating u* t,s) is

$$E\left\{[u^*(t,s) - u(t,s|T)]^2\right\} = \phi^T(s)P(t|T)\phi(s) \tag{4.27}$$

It is interesting to investigate the overall performance measure $\hat{J}$. From the error bound result in Section 3.6, and from (4.15), the following inequality is obtained

$$\hat{J} \le \int_n^\infty f(x')\,dx' + \sum_{j=1}^{n} p_{j'}(t) \tag{4.28}$$

where f(x') is a monotone decreasing function of x' such that

$$f(j) = p_{2j-1,2j-1}(0) + 2p_{2j-1,2j}(0) + p_{2j,2j}(0) + M'/2\omega_j^2c^2 \tag{4.29}$$

for heat problems, or

$$f(j) = p_{2j-1,2j-1}(0) + 2p_{2j-1,2j}(0) + p_{2j,2j}(0) + M'/2\omega^2c^2 \qquad (4.30)$$

for wave problems. Where M is defined by (3.128) and M' is defined by

$$M' = \text{lub}\ [c^4(\ q_{00}(t)\phi_j^2(0) + q_{01}(t)\phi_j(0)\phi_j(1)$$

$$+ q_{10}(t)\phi_j(0)\phi_j(1) + q_{11}(t)\phi_j^2(1)) + Q(t)] \qquad (4.31)$$

$$j = n,\ n+1,\ n+2,\ .\ .\ .\ .$$

for the initial-boundary-value wave problem which was not discussed in Section 3.6.

For the example given in Section 3.6.1, J becomes

$$J \leq \frac{2k + M}{k\pi^2(2n+1)} + \text{tr}\{P\} \qquad (4.32)$$

which gives the bound of uncertainty caused by the finite modal representation error plus the estimation error. Note that with the Kalman filter, the covariance P(t) can be solved for a priori. That is, (4.32) can be obtained before any measurement is made or processed, at all. In a number of applications, this value along with the cost of computation, computational noise values, and the modeling of observation processes plays an important role in deciding how many modes should be considered, i.e., if the computational noise is modeled and is added to the right side of (4.32) it can be rewritten as

$$J \leq \frac{2k + M}{k\pi^2(2n+1)} + tr\{P\} + \sum_{j=1}^{n} p_j^c \tag{4.33}$$

where p_j^c represents the computational noise associated with the j-th mode. By taking n+1 mode into consideration the decrease in uncertainty becomes

$$\frac{2k + M}{k\pi^2} \left(\frac{1}{2n+1} - \frac{1}{2(n+1)+1} \right) = \frac{2(2k+M)}{k\pi^2(2n+1)(2n+3)} \tag{4.34}$$

and the increase of uncertainty is

$$p_{n+1,n+1} + p_{n+1}^c \tag{4.35}$$

where $p_{n+1,n+1}$ is the (n+1)th diagonal element of P and $p^c{}_{n+1}$ is the computational error associated with the (n+1)th mode, respectively. The logical choice of n has to satisfy the inequality

$$\frac{2(2k + M)}{k\pi^2(2n+1)(2n+3)} > p_{n+1,n+1} + p_{n+1}^c \tag{4.36}$$

which can be rewritten as

$$4n^2 + 8n - (\frac{2(2k + M)}{k\pi^2(p_{n+1,n+1} + p^c_{n+1})} - 3) < 0 \tag{4.37}$$

Solving (4.36) for n and knowing that n is a positive integer, one obtains

$$n < 1 + \frac{(4 + k')^{1/2}}{2} \tag{4.38}$$

where $k'=2(2k+M)/k\pi^2(p_{n+1,n+1}+p^c_{n+1})-3$. The largest integer less than the right side of (4.37) is the maximum number of modes which should be carried without introducing more uncertainty resulting from the estimation algorithm and the computation. As discussed previously, (4.28) can be obtained before any measurement is made or processed. This enables one to decide the number of modes to be considered in the application of the algorithm when the problem involves an infinite number of modes.

In the heat problems and wave problems proposed in this study, the matrices A, C, and D, are time invariant, hence it is of interest to investigate steady state results when Q and K are also given as constants. After the filter given by (4.17) through (4.19) has been in operation a long time then the filtering process may reach a steady state in the sense that P becomes a constant matrix ($\dot{P}=0$). In principle, this steady state matrix may be obtained by solving the $\frac{1}{2}n(n+1)$ simultaneous quadratic equations given by setting $\dot{P}=0$ in (4.19):

$$0 = AP + PA^T + DQD^T - PC^TR^{-1}CP \tag{4.39}$$

In this steady state, the rate at which uncertainty builds indicated by the term, DQD^T, is just balanced by the rate at which new information comes into the system, indicated by $PC^TR^{-1}CP$, and by any damping the system may have (as expressed in A) [33]. In practice, the solution of (4.39) for P is impracticable for $n>2$; instead, (4.19) is integrated, say, with $P(0)=0$, until $\dot{P}\to 0$. In the steady state case referred to here, one then obtains a single number for the performance bound indicated by (4.28), i.e., the bound is not a function of time.

4.3. State Estimation for Spatial Dynamics

When the temporal dependence of the random field is prescribed by a set of orthonormal coordinate functions, the resultant message models are the system of stochastic ordinary differential equations with spatial parameters s, described by (3.194) to (3.202) for a random wave process. The second order temporal differential operator appearing in wave processes made it feasible to consider the decomposition of the wave process into a set of spatial equations evolving along each temporal basis function. However, a realistic measurement process is assumed to be a continuous function in time at discrete locations in space. This particular feature of the problem suggests the development of an estimation algorithm which has not been considered within the scope of conventional estimation theory. There are several different formulations of the problem depending on the assumptions imposed on each wave process.

4.3.1 Ordered Sequential Formulation

As discussed in the previous section, the error criterion in the sense of MMSE is formed as

$$\hat{J} \triangleq E\left\{\|u(t,s_{i+1}) - \hat{u}(t,s_{i+1})\|\right\} \quad i = 0,\, 1,\, \ldots\, m \tag{4.40}$$

where the norm $\|\cdot\|$ is defined in $L^2(0,T)$. The subscripts indicate that only discrete spatial points are considered. Minimizing the above expression to find the optimum estimate of the state $u(t,s_{i+1})$ given the message model described by (3.43) to (3.46) and the continuous-time discrete-space measurements described by (4.1) to (4.3) yields the two-fold problem of modal approximation and estimation. Since the modal representation is obtained by (3.194) to (3.202), the error criteria (4.40) can be rewritten as

$$\hat{J} = E\left\{\sum_{j=n+1}^{\infty} x_{2j-1}^2(s_{i+1}) + \sum_{j=1}^{n}[x_{2j-1}(s_{i+1}) - \hat{x}_{2j-1}(s_{i+1})]^2\right\} \tag{4.41}$$

Interchanging expectation and summation gives

$$\hat{J} = \sum_{j=n+1}^{\infty} E\left\{x_{2j-1}^2(s_{i+1})\right\} +$$

$$\sum_{j=1}^{n} E\left\{[x_{2j-1}(s_{i+1}) - \hat{x}_{2j-1}(s_{i+1})]^2\right\} \tag{4.42}$$

By letting

$$P(s_{i+1}) = E\left\{[x(s_{i+1}) - \hat{x}(s_{i+1})][x(s_{i+1}) - \hat{x}(s_{i+1})]^T\right\} \tag{4.43}$$

(4.42) becomes

$$\hat{J} = \sum_{j=n+1}^{\infty} E\left\{x_{2j-1}^2(s_{i+1})\right\} + \sum_{j=1}^{n} p_{2j-1,2j-1}(s_{i+1}) \tag{4.44}$$

In view of the discrete-spatial measurements, the message model given by (3.94) through (3.202) is to be discretized with spatial interval d . Discretizing the message model gives

$$x(i+1) = \Phi(i)x(i) + \bar{w}(i) \tag{4.45}$$

The initial spatial statistics are

$$E\{x(0)\} = \mu_0 \tag{4.46}$$

$$E\left\{[x(0) - \mu_0][x(0) - \mu_0]^T\right\} = P_0 \tag{4.47}$$

and the noise statistics are

$$E\{\bar{w}(i)\} = 0 \tag{4.48}$$

$$E\left\{\bar{w}(i)\bar{w}^T(j)\right\} = Q(i)\,\delta_{ij} \qquad i, j = 1, 2, \ldots m \tag{4.49}$$

The terms $\Phi(i)$, $w(i)$ and $Q(i)$ are defined as

$$\Phi(i) = \Phi(s_{i+1}, s_i)$$

$$\triangleq \text{diag}\left[\begin{array}{cc|cc|c|cc} \cos\frac{\omega_1 d}{c} & \sin\frac{\omega_1 d}{c} & \cos\frac{\omega_2 d}{c} & \sin\frac{\omega_2 d}{c} & & \cos\frac{\omega_n d}{c} & \sin\frac{\omega_n d}{c} \\ -\sin\frac{\omega_1 d}{c} & \cos\frac{\omega_1 d}{c} & -\sin\frac{\omega_2 d}{c} & \cos\frac{\omega_2 d}{c} & \cdots & -\sin\frac{\omega_n d}{c} & \cos\frac{\omega_n d}{c} \end{array}\right] \tag{4.50}$$

$$w(i) \triangleq \int_{s_i}^{s_{i+1}} \Phi(s_{i+1}, s')Dw(s')\, ds' \tag{4.51}$$

and

$$Q(i) \triangleq \int_{s_i}^{s_{i+1}} \Phi(s_{i+1}, s')DQ(s')D^T\Phi^T(s_{i+1}, s')\, ds' \tag{4.52}$$

where

$$d \triangleq s_{i+1} - s_i \qquad i = 0, 1, 2, \ldots m \tag{4.53}$$

Note that the discretization of the message model is convenient because the observation process is taken in continuous time at equally spaced spatial location. At stage j the observation set is described by

$$z(t,i) = u(t,i) + v(t,i) \qquad 0 \leq t \leq T \ ; \ i = 1, 2, \dots m \tag{4.54}$$

where $\{v(t,i)\}$ is zero mean white noise with

$$E\{v(t,i)v(t',k)\} = r(i)\delta(t-t')\delta_{ik} \tag{4.55}$$

It is assumed that $Q(i)>0$ and $r(i)>0$ and that $\{w(i)\}$, $\{v(t,i)\}$, and $x(0)$ are not correlated with each other. The filter structure is constrained to be of the linear form

$$x(i+1) = F(i)x(i) + K(i+1)[\int_0^T m(t',i+1)z(t',i+1)\, dt'] \tag{4.56}$$

where the sequence of 2n x 2n matrices $\{F(i)\}$, and 2n x n matrices $\{K(i+1)\}$, the initial condition x_0, and the nx1 vector weighting function $m(t,i+1)$ are to be selected in such a way as to have an unbiased estimate at each stage, and to minimize the mean-square error at each stage. The term in the bracket of (4.51) can be regarded as a temporal preprocessing of the observation set to yield an equivalent spatial measurement sequences

$$\hat{z}(T,i+1) = \int_0^T m(t',i+1)\left[\sum_{j=1}^{\infty} \phi_j(t')x_j(i+1)\right] dt'$$

$$+ \int_0^T m(t',i+1)v(t',i+1)\, dt' \qquad 0 \le t \le T \qquad (4.57)$$

which will drive the spatial linear filter. If the set of weighting $m(t,i+1)$ is chosen to maximize the signal-to-noise ratio of the output of the preprocessor, this will minimize the mean-square-error of the optimum linear spatial filter. For each preprocessor the output due to the signal $x_k(i+1)$ is a random variable modulated by a deterministic quantity:

$$s_k(T,i+1) = x_k(i+1)\int_0^T m_k(t',i+1)\phi_k(t')\, dt' \qquad (4.58)$$

and the output due to the noise $v(t,j+1)$ and other signals $\sum_{j=1}^{\infty} x_j(i+1) - x_k(i+1)$ are also random variables:

$$n_k(t,i+1) = \sum_{\substack{j=1 \\ j\neq k}}^{\infty} x_j(i+1)\int_0^T m_k(t',i+1)\phi_j(t')\, dt'$$

$$+ \int_0^T m_k(t',i+1)v(t',i+1)\, dt' \qquad k = 1, 2, \ldots n \qquad (4.59)$$

Now the output signal-to-noise ratio at time T for modal coefficient k is defined as

$$\left(\frac{S}{N}\right)_k \triangleq E\left\{\frac{s_k^2(T,i+1)}{n_k^2(t,i+1)}\right\} \tag{4.60}$$

Substituting (4.95) and (4.96) into (4.97) yields

$$\left(\frac{S}{N}\right)_k = \frac{E\left\{x_k^2(i+1)\right\}[\int_0^T m_k(t,i+1)\phi_k(t)\ dt]^2}{\sum_{\substack{j=1\\j\neq k}}^{\infty} E\left\{x_j^2(i+1)\right\}[\int_0^T m_k(t,i+1)\phi_j(t)\ dt]^2 \cdots}$$

$$\cdots \overline{+\ E\left\{\int_0^T dt \int_0^T m_k(t,i+1)m_k(t',i+1)v(t,i+1)v(t',i+1)dt'\right\}} \tag{4.61}$$

Interchanging the expectation and the integration operation and using the sifting property of the delta function for the second term of the denominator, one obtains

$$\left(\frac{S}{N}\right)_k = \frac{E\left\{x_k^2(i+1)\right\}}{\sum_{\substack{j=1\\j\neq k}}^{\infty} E\left\{x^2(i+1)\right\}\left[\frac{\int_0^T m_k(t,i+1)\phi_j(t)dt}{\Gamma_k}\right]^2 + \frac{r(i+1)\int_0^T m_k^2(t,i+1)\ dt}{\Gamma_k^2}} \tag{4.62}$$

where Γ_k is defined as

$$\Gamma_k = \int_0^T m_k(t,i+1)\phi_k(t)\ dt$$

for the k-th preprocessor at each stage j+1 where $k = 1, 2, \ldots n$. Since $E\{x_j^{\,2}(i+1)\}$ and $r(i+1)$ are positive definite, minimization of the denominator with respect to $m_k(t,i+1)$ to maximize the signal-to-noise ratio is the problem of interest. The solution follows by applying the Schwartz Inequality, i.e.,

$$\left[\int_0^T f(t)g(t)\,dt\right]^2 \le \int_0^T f^2(t)\,dt \int_0^T g^2(t)\,dt \tag{4.63}$$

to the second term of the denominator. This term can be rewritten in the form

$$\frac{\left[\int_0^T m_k(t,i+1)\phi_k(t)\,dt\right]^2}{r(i+1)\int_0^T m_k^2(t,i+1)\,dt} \tag{4.64}$$

and is to be maximized. If $\int_0^T f^2(t)\,dt \neq 0$, (4.63) can be rearranged as

$$\frac{\left[\int_0^T f(t)g(t)\,dt\right]^2}{\int_0^T f^2(t)\,dt} \le \int_0^T g^2(t)\,dt \tag{4.65}$$

This implies that (4.64) is bounded

$$\frac{[\int_0^T m_k(t,i+1)\phi_k(t)\ dt]^2}{r(i+1)\int_0^T m_k^2(t,i+1)\ dt} \leq \frac{1}{r(i+1)} \int_0^T \phi_k^2(t)\ dt = \frac{1}{r(i+1)} \tag{4.66}$$

where the property of the orthonormal set $\{\phi_j(t)\}$, i.e.,

$$\int_0^T \phi_j(t)\phi_k(t)\ dt = \delta_{jk} \tag{4.67}$$

is used. From (4.66) it is obvious that equality holds if an only if

$$m_k(t,i+1) = b\phi_k(t) \tag{4.68}$$

for every stage i+1 where the value of b does not affect the performance measure, and can be adjusted to give a prescribed average gain over the interval [0,T]. The minimization of the first term in the denominator of (4.62) is also achieved by the result obtained for the second term. This is due to the fact that the property of the orthonormal set given by (4.67) yields the value zero, the smallest value possible, to all terms in the summation.

Substituting (4.68) into (4.57) gives the expression of the measurement sequence preprocessed in time:

$$z(T,i+1) = \begin{bmatrix} x_1(i+1) \\ x_2(i+1) \\ \cdot \\ \cdot \\ \cdot \\ x_{2n-1}(i+1) \end{bmatrix} + \begin{bmatrix} v_1(i+1) \\ v_2(i+1) \\ \cdot \\ \cdot \\ \cdot \\ v_n(i+1) \end{bmatrix} \tag{4.69}$$

or

$$z(T,i+1) = C(i+1)x(i+1) + v(i+1)$$

$$i = 1, 2, \ldots m \tag{4.70}$$

where

$$C = [1 \quad 0 \quad 1 \quad 0 \quad \ldots \quad 1 \quad 0]_{1\times 2n} \tag{4.71}$$

The constant b is prescribed to be one and v_j is given by

$$v_j(i+1) = \int_0^T \phi_j(t)v(t,i+1)\, dt$$

$$j = 1, 2, \ldots n \quad i = 1, 2, \ldots m \tag{4.72}$$

The statistical properties are given as

$$E\{v(i)\} = 0 \tag{4.73}$$

$$E\{v(i)v\ (j)\} = R(i)\delta_{ij}$$

$$i, j = 1, 2, \ldots m \tag{4.74}$$

where

$$R(i) = \begin{bmatrix} r(i) & & & 0 \\ & r(i) & & \\ & & \ddots & \\ 0 & & & r(i) \end{bmatrix} \tag{4.75}$$

And the resulting filter structure becomes

$$\hat{x}(i+1) = F(i)\hat{x}(i) + K(i+1)\hat{z}(T,i+1) \tag{4.76}$$

where the sequence of 2n x 2n matrix F(i), 2n x n matrix K(i+1), and the initial condition x_0 are chosen as discussed in Sage [49]. For an unbiased estimate it is sufficient that

$$\hat{x}_0 = E\{x_0\} \tag{4.77}$$

and

$$F(i) = [I - K(i+1)C(i+1)]\Phi(i) \qquad i = 1, 2, \ldots m \tag{4.78}$$

Substituting (4.78) in (4.76) results in a filter structure established in more detail,

$$\hat{x}(i+1) = \Phi(i)\hat{x}(i) + K(i+1)[\hat{z}(T,i+1) - C(i+1)\Phi(i)\hat{x}(i)] \tag{4.79}$$

The error associated with the filter described by (4.79) propagates according to the difference equation,

$$e(i+1) = [I - K(i+1)C(i+1)]\Phi(i)e(i) + [I - K(i+1)C(i+1)]\bar{w}(i) - K(i+1)v(i+1) \tag{4.80}$$

and the expression describing the propagation of the error variance for such an equation is known [49], i.e., with $P(i)=E\{e(i)e^T(i)\}$ it is seen that

$$P(i+1) = [I - K(i+1)C(i+1)]M(i)[I - K(i+1)C(i+1)]^T + K(i+1)R(i+1)K^T(i+1) \tag{4.81}$$

where

$$M(i) \triangleq \Phi(i)P(i)\Phi^T(i) + Q(i) \tag{4.82}$$

The initial condition for (4.81) is $P(0)=\mathrm{Var}\{x(0)\}$. In order to select $K(j+1)$ in such a way as to minimize the mean-square error the performance measure is formed as

$$E\{e^T(i+1)e(i+1)\} = \mathrm{tr}\{P(i+1)\} \tag{4.83}$$

The minimization of (4.83) with respect to K(i+1) is easily accomplished, leading to the expression

$$K(i+1) = M(i)C(i+1)[R(i+1) + C(i+1)M(i)C^T(i+1)]^{-1} \tag{4.84}$$

Substituting (4.84) in (4.81) indicates that P(i+1) may be expressed as

$$P(i+1) = M(i) - M(i)C^T(i+1)[R(i+1) + C(i+1)M(i)C^T(i+1)]^{-1}C(i+1)M(i) \tag{4.85}$$

The algorithm is thus established. Note that the stationary statistical properties assumed in time made it possible to make use of the Schwartz Inequality in finding the vector weighting function m(t,i). In the previous work [4], the scalar problem with non-stationary measurement noise is solved by utilizing a single stage optimization procedure. The filtered estimate of u*(t,i) is

$$\hat{u}(t,i) = \sum_{j=1}^{n} \phi_j(t)\hat{x}_{2j-1}(i) \tag{4.86}$$

and the error variance of the estimate $\hat{u}(t,i)$ becomes

$$E\left\{[u^*(t,i) - \hat{u}(t,i)]^2\right\}$$

$$= \sum_{j=1}^{n} \phi_j^2(t)p_{2j-1,2j-1}(i) \tag{4.87}$$

The filtered estimate $\hat{x}(i)$ and the filtering error covariance matrix $P(i)$ are discontinuous at each discrete point where the temporal measurement processes are available. However, the smoothed estimate $\hat{x}(s|s_f)$, obtained by applying the discrete-continuous filter, described by (2.81) to (2.88), and the fixed interval smoothing algorithm, (2.89) through (2.94), with the change of independent variable from t to s, is continuous. The resulting smoothed estimate of $u^*(t,s)$ becomes

$$\hat{u}(t,s|s_f) = \sum_{j=1}^{n} \phi_j(t)\hat{x}_{2j-1}(s|s_f) \tag{4.88}$$

where the finite observation interval $[0,s_f]$ is assumed and the corresponding smoothing error variance is

$$\begin{aligned} E\left\{[u^*(t,s) - \hat{u}(t,s|s_f)]^2\right\} &= \sum_{j=1}^{n} \phi_j^2(t)\mathrm{var}\{\tilde{x}_{2j-1}(s|s_f)\} \\ &= \sum_{j=1}^{n} \phi_j^2(t)p_{2j-1,2j-1}(s|s_f) \end{aligned} \tag{4.89}$$

The smoothing error variance of j-th mode, $p_{2j-1,2j-1}(s|s_f)$ is the diagonal element of the matrix $P(s|s_f)$ given by (2.94). As shown in (2.93), the smoothing gain may be obtained from the knowledge of filtering error variance, which allows one not to compute the smoothing error variance unless one is interested in the performance of the smoother.

The overall performance measure resulting from the modal representation and the estimation procedure takes the same form as given in Section 4.2.2 exept for the fact that the smoothing error variance is utilized and the independent parameter is switched from t to s. That is,

$$\hat{J} \leq \int_{n}^{\infty} f(x')\, dx' + \sum_{j=1}^{n} p_{2j-1,2j-1}(s|s_f) \tag{4.90}$$

where f(x') is defined by (3.231) and (3.222).

In this section the measurement noise is assumed to be stationary for the derivation of the algorithm. The non-stationary case is studied with a different measurement model in the next section.

4.3.2 Optimal Estimation for Gaussian Systems [51]

Consider the discrete spatial message model given by (4.45) through (4.53) and the observation set described by

$$z(t,i) = C(t,i)x(t,i) + v(t,i) \tag{4.91}$$

where the statistical properties of the measurement noise v(t,i) are assumed to be

$$E\{v(t,i)\} = 0 \tag{4.92}$$

$$E\{v(t,i)v(t',j)\} = r(t,i)\delta(t-t')\delta_{ij} \tag{4.93}$$

and w(i), v(t,i) and x(0) are Gaussian but independent processes. The measurement matrix C(t,i) is of the form

$$C(t,i) = [\phi_1(t) \;\; 0 \;\; \phi_2(t) \;\; 0 \;\; . \;.\; . \;\; \phi_n(t) \;\; 0] \tag{4.94}$$

which implies that the finite number n is chosen such that the measurement process can be modeled as (4.91) for practical purposes. The optimal conditional mean estimate of x(i) is to be investigated when the observation noise is a Gaussian non-stationary process.

The measurement set at a stage i is denoted as

$$\bar{z}(i) = \left\{ z(t,i) \; : \; t \in [0,T] \right\} \tag{4.95}$$

and the set of measurements available up to that stage as

$$Z(i) = \left\{ \bar{z}(1), \, \bar{z}(2), \, . \; . \; . \; , \, \bar{z}(i) \right\} \tag{4.96}$$

The problem is to find the conditional mean estimate

$$\hat{x}(i\,|k) = E\{x(i)\,|Z(k)\} \tag{4.97}$$

In (4.97) if k<i one has the prediction problem, if k=i one has the filtering problem, and if k=m, one has the fixed interval smoothing problem. The latter problem is of primary importance in a practical situation,

since in a spatial sense, causality is not usually a requirement.

Suppose at stage i the conditional mean and variance,

$$\hat{x}(i|i-1) = E\{x(i)|Z(i-1)\} \tag{4.98}$$

and

$$P(i|i-1) = Var\{x(i)|Z(i-1)\} \tag{4.99}$$

are known. Then using the well known continuous time Kalman filter results, these terms, (4.98) and (4.99), can be easily updated, i.e.,

$$\hat{x}(i|i) = E\{x(i)|Z(i)\} = \hat{x}(T,i) \tag{4.100}$$

where x(T,i) is generated by integrating the filter equation

$$\dot{\hat{x}}(t,i) = K(t,i)[z(t,i) - C(t,i)\hat{x}(t,i)] \tag{4.101}$$

up to time t=T. The initial condition for (4.101) is

$$\hat{x}(0,i) = \hat{x}(i|i-1) = E\{x(i)|Z(i-1)\} \tag{4.102}$$

The gain in (4.101) is evaluated according to

$$K(t,i) = P(t,i)C^T(t,i)r^{-1}(t,i) \tag{4.103}$$

where

$$\dot{P}(t,i) = - P(t,i)C^T(t,i)r^{-1}(t,i)C(t,i)P(t,i) \tag{4.104}$$

is solved from the initial condition (4.99),

$$P(0,i) = P(i|i-1) = \mathrm{Var}\left\{x(i)|Z(i-1)\right\} \tag{4.105}$$

Note that

$$P(i|i) = \mathrm{Var}\left\{x(i)|Z(i)\right\} = P(T,i) \tag{4.106}$$

From (4.45) the one stage predicted mean and variance is easily obtained, i.e.,

$$\hat{x}(i+1|i) = E\left\{x(i+1)|Z(i)\right\} = \Phi(i)\hat{x}(T,i) \tag{4.107}$$

$$\begin{aligned} P(i+1|i) &= \mathrm{Var}\left\{x(i+1)|Z(i)\right\} \\ &= \Phi(i)P(T,i)\Phi^T(i) + D(i)Q(i)D^T(i) \end{aligned} \tag{4.108}$$

The filtering algorithm for generating the conditional mean and variance is thus established. The algorithm is initiated from the conditions

$$\hat{x}(0|0) = \mu_0 = \hat{x}(T,0) \tag{4.109}$$

$$P(0|0) = P_0 = P(T,0) \tag{4.110}$$

The resulting filtered estimate of u*(t,s) and the corresponding filtering error variance are

$$\hat{u}(t,s_i) = \sum_{j=1}^{n} \phi_j(t)\hat{x}_{2j-1}(i|i) \tag{4.111}$$

and

$$E\left\{[u^*(t,s) - \hat{u}(t,s)]^2\right\}$$

$$= \sum_{j=1}^{n} \phi_j^2(t)p_{2j-1,2j-1}(i|i) \tag{4.112}$$

To develop the optimal smoothed estimate it is helpful to realize that the conditional mean and variance given by (4.98) and (4.106) are all that is necessary to be stored, and that one need not store all the temporal data sets. As a first step it is convenient to develop an idea of equivalent discrete observations of time series in space. One may then apply the standard discrete fixed interval smoothing algorithm to obtain the conditional mean estimate given Z_m, $E\{x(i)|Z_m\}$. Suppose that at stage i one has processed the observation set z(t,i) optimally and reduced the error variance accordingly, so that by application of (4.104) the new data set takes one from a variance of P(0,i) to a value P(T,i). (4.104) is a type of Riccati equation that may be explicitly solved. The solution to this linear equation is

$$P(T,i) = [P^{-1}(0,i) + \int_0^T C^T(t,i)r^{-1}(t,i)C(t,i)\, dt]^{-1} \tag{4.113}$$

One seeks a single discrete measurement of the form

$$Y(i) = H(i)x(i) + v(i) \tag{4.114}$$

which would have precisely the same effect on the error variance, i.e., bring it from a value P(0,i) to a value P(T,i). The discrete measurement noise is zero mean independent white Gaussian noise with covariance $R_D(i)$. From the basic theory of Gaussian random variables [50], it is known that

$$\begin{aligned} \mathrm{Var}\{x(i)|\, Z(i-1), y(i)\} &= \mathrm{Var}\{x(i)|\, Z(i-1)\} \\ &\quad - \mathrm{Cov}\{x(i), y(i)|\, Z(i-1)\}\mathrm{Var}^{-1}\{y(i)|\, Z(i-1)\}\cdot \\ &\quad \mathrm{Cov}\{x(i), y(i)|\, Z(i-1)\} \end{aligned} \tag{4.115}$$

or symbolically

$$P(i|i) = P(i|i-1) - P_{xy}(i|i-1)P_{yy}^{-1}(i|i-1)P_{yx}(i|i-1)$$

Using (4.114) then gives

$$P(i|i) = P(i|i-1) - P(i|i-1)H^T(i)[H(i)P(i|i-1)H^T(i) + R_D(i)]^{-1} \cdot H(i)P(i|i-1) \qquad (4.116)$$

If one applies the matrix inversion lemma [49], (4.116) can be written as

$$P(i|i) = [P^{-1}(i|i-1) + H^T(i)R_D^{-1}(i)H(i)]^{-1} \qquad (4.117)$$

Since $P(i|j)$ is to be equivalent to $P(T,i)$ and since $P(i|i-1)$ is to be equivalent to $P(0.i)$ one can rewrite (4.117) accordingly

$$P(T,i) = [P^{-1}(0,i) + H^T(i)R_D^{-1}(i)H(i)]^{-1} \qquad (4.118)$$

Equating (4.118) with (4.113), it is seen that a single measurement of the form indicated by (4.110) is equivalent to a continuous measurement of the type given by (4.54), provided that

$$H^T(i)R_D^{-1}(i)H(i) = \int_0^T C^T(t,i)R^{-1}(t,i)C(t,i)\,dt \qquad (4.119)$$

Thus, there exists an equivalent discrete measurement which is not unique and the fixed interval smoothing algorithm as presented in Section 2.4.2 can be used to obtain the conditional mean estimate $\hat{x}(i|m)$. The process of filtering forward in space and then smoothing backwards spatially is an efficient means of generating $\hat{x}(i|m)$ for $i = 0, 1, \ldots m$. The sequential procedure described here is computationally preferable to the obvious method of defining a large state vector composed of all the component vectors, $x(i)$, and then using an extremely high order temporal Kalman filter. The filtered estimate of $u^*(t,i)$, the corresponding error variance and the overall performance bound can be obtained as discussed in the previous sections. It is of interest to observe that the algorithm developed herein is optimum when the observation model defined by (4.91) holds and applies to a system with a non-stationary observation process while the previous algorithm furnishes a simpler algorithm for stationary systems.

4.4 Illustrative Examples

In this Section two examples will be considered in detail to demonstrate the estimation algorithms described, and the performance measure studied in the previous Sections. The first example to be considered is the one dimensional random heat equation discussed in Section 3.4.3. The method of decomposition into a set of stochastic ordinary differential equations is used by finding a set of spatial basis functions. The state estimation technique for a temporal system is applied to the example.

The second problem to be investigated is the spatial dimensioned random wave equation discussed in Section 3.5.2. For this example a finite number of modes results from the decomposition. The use of the algorithms studied in Sections 4.3.1 and 4.3.2 is demonstrated.

These highly simplified examples are of some practical value. The systems do exhibit important characteristics found in random D. P. systems describing heat and wave processes. For that reason, it is a good testing ground for the techniques that have been developed.

4.4.1 Heat Example

Consider the modal representation of the heat system, (3.102) to (3.107), obtained for an illustrative example in Section 3.4. The first two modes are taken into consideration to yield the state vector form

$$\dot{x}(t) = \begin{bmatrix} -k(3\pi/2)^2 & 0 \\ 0 & -k(5\pi/2)^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -\sqrt{2k} & 1 & 0 \\ -\sqrt{2k} & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ w_1(t) \\ w_2(t) \end{bmatrix} \tag{4.120}$$

where

$$E\{x(0)\} = 0 \tag{4.121}$$

$$E\left\{x(0)x^T(0)\right\} = \begin{bmatrix} \dfrac{4\bar{p}_0}{9\pi^2} & 0 \\ 0 & \dfrac{4\bar{p}_0}{25\pi^2} \end{bmatrix} \tag{4.122}$$

$$E\{\bar{w}(t)\} = E\left\{[v_1(t) \quad w_1(t) \quad w_2(t)]^T\right\} = 0 \tag{4.123}$$

and

$$E\{\bar{w}(t)\bar{w}(t')\} = \bar{Q}\delta(t-t')$$

$$= \begin{bmatrix} q_{11} & & 0 \\ & q_1 & \\ 0 & & q_2 \end{bmatrix} \delta(t-t') \tag{4.124}$$

The measurements taken by four point sensors imbedded in the spatial domain are described by

$$z(t) = \begin{bmatrix} \phi_1(s_1) & \phi_2(s_1) \\ \phi_1(s_2) & \phi_2(s_2) \\ \phi_1(s_3) & \phi_2(s_3) \\ \phi_1(s_4) & \phi_2(s_4) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ v_4(t) \end{bmatrix} \tag{4.125}$$

where the white Gaussian measurement noise v(t) is characterized by

$$E\{v(t)\} = 0 \tag{4.126}$$

$$E\left\{v(t)v^T(t')\right\} = R\,\delta(t-t') \tag{4.127}$$

The filtering equations (4.17) through (4.21) solved with the values

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad E\{x(0)\} = 0 \tag{4.128}$$

$$P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix} \qquad P_0 = \begin{bmatrix} \frac{4\bar{p}_0}{9\pi^2} & 0 \\ 0 & \frac{4\bar{p}_0}{25\pi^2} \end{bmatrix} \tag{4.129}$$

$$A = \begin{bmatrix} -k(3\pi/2)^2 & 0 \\ 0 & -k(5\pi/2)^2 \end{bmatrix} \qquad D = \begin{bmatrix} -2k & 1 & 0 \\ -2k & 0 & 1 \end{bmatrix} \tag{4.130}$$

$$C = \begin{bmatrix} \phi_1(s_1) & \phi_2(s_1) \\ \phi_1(s_2) & \phi_2(s_2) \\ \phi_1(s_3) & \phi_2(s_3) \\ \phi_1(s_4) & \phi_2(s_4) \end{bmatrix} \qquad R = \begin{bmatrix} r & & & 0 \\ & r & & \\ & & r & \\ 0 & & & r \end{bmatrix} \tag{4.131}$$

$$K(t) = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} \tag{4.132}$$

The filtered estimate of $u^*(t,s)$, $\hat{u}(t,s)$, and the error variance of $\hat{u}(t,s)$ are found using (4.22) and (4.23). The performance measure of the overall modal estimation procedure can be obtained from (4.28) as

$$\hat{J} = \sum_{j=3}^{\infty} E\left\{x_j^2(t)\right\} + \operatorname{tr}\left\{P(t)\right\} \tag{4.133}$$

which yields the error bound of the form

$$\hat{J} \leq \frac{2kp_0 + M}{5k\pi^2} + \operatorname{tr}\left\{P(t)\right\} \tag{4.134}$$

and implies that the energy in the error caused from taking two modes only is bounded by an explicit number.

A computer simulation was conducted with the following parameter values:

$$k = 0.05 \qquad r = 0.1$$

$$0 \leq t \leq 1.0 \qquad 0 \leq s \leq 1.0$$

$$\bar{p}_0 = 18.0 \qquad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 0.333 \\ 0.750 \\ 0.800 \\ 1.000 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

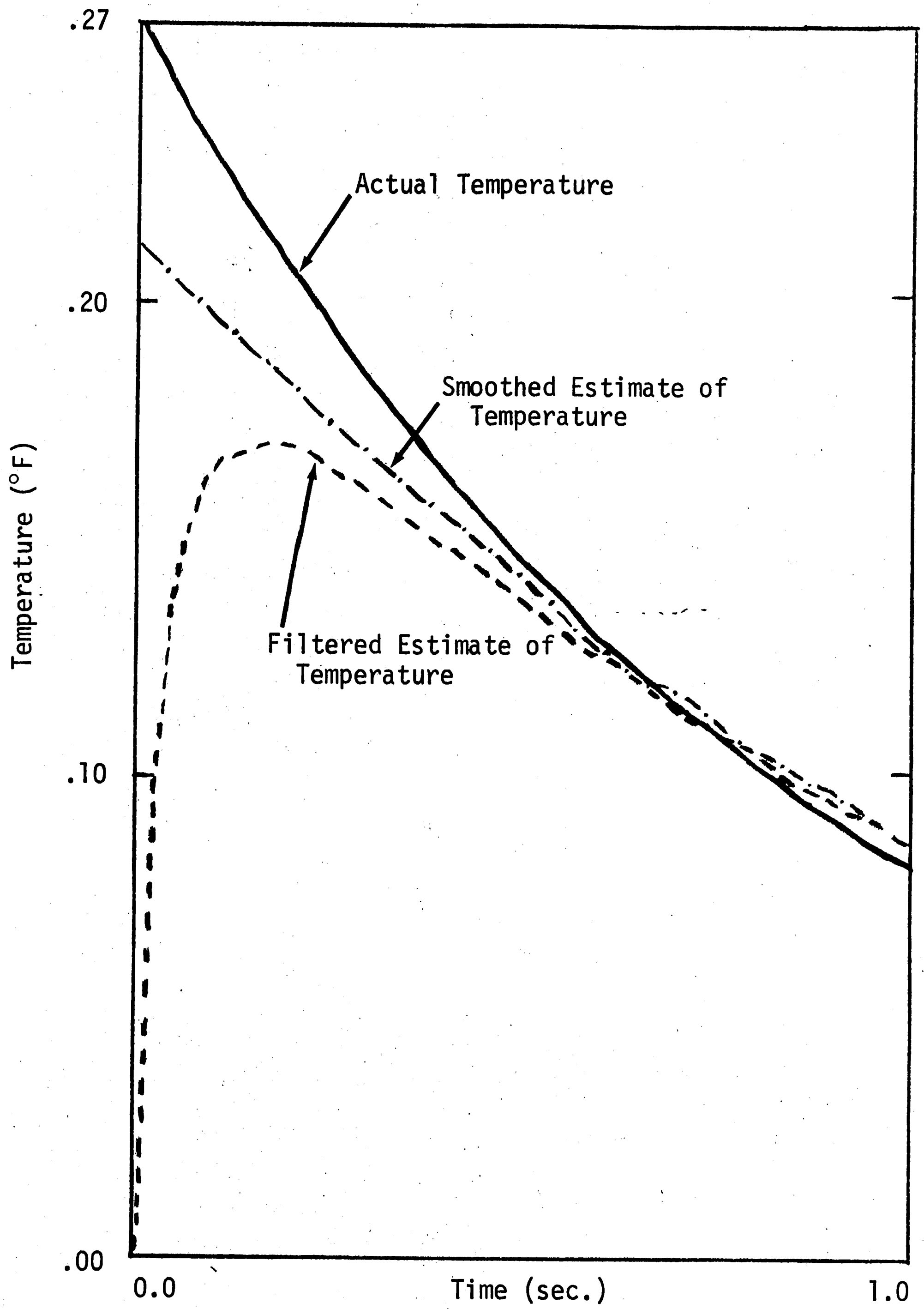


Figure 1. Actual and Estimated Value of u(t,s) for Heat Problem

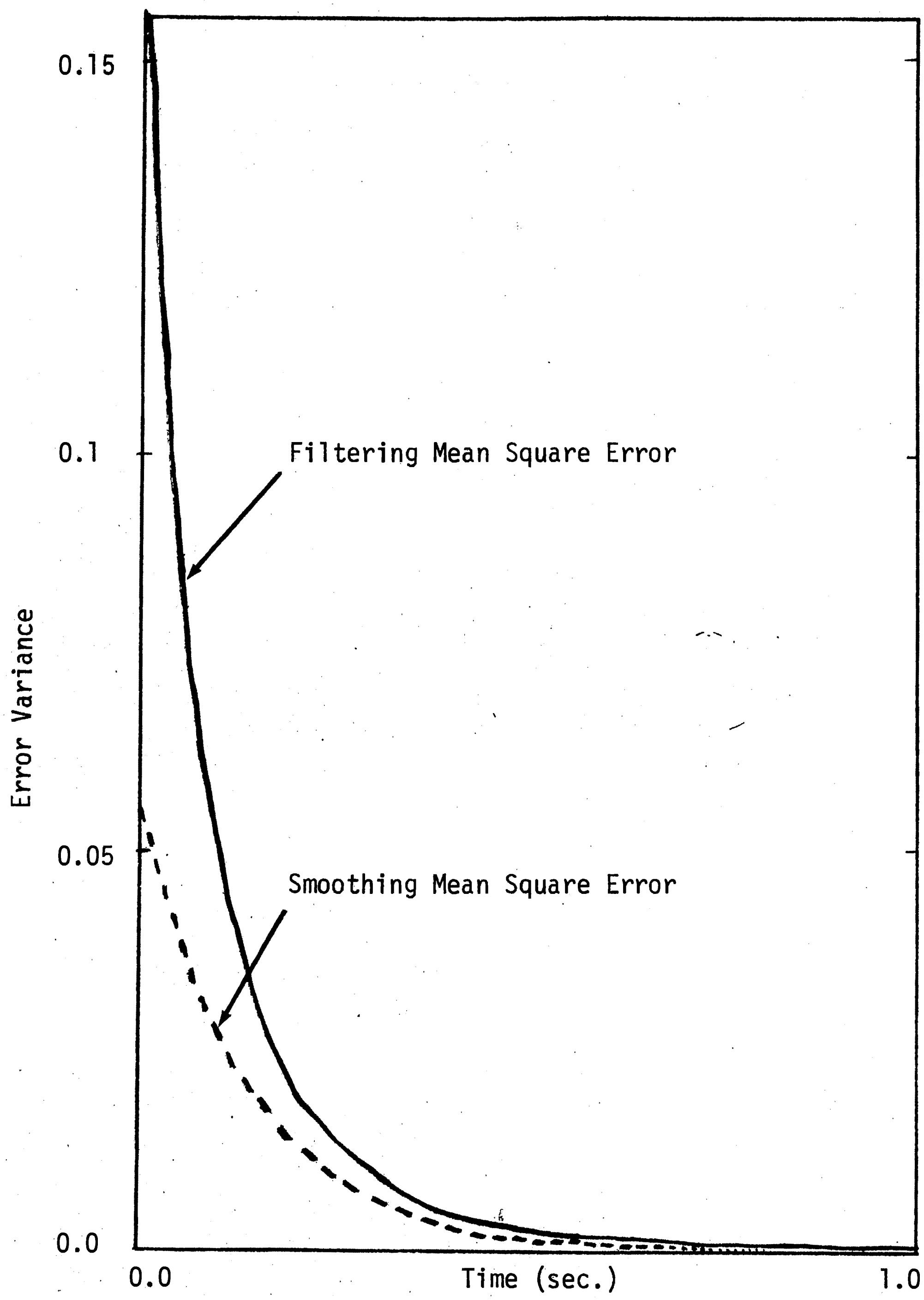


Figure 2. Mean Square Error for Heat Problem

The results of the experiment are shown in Figures 1 and 2 where the actual and the estimated values of the state $u(t,s)$ at $s = 0.8$ are indicated with the corresponding mean square error.

4.4.2 Wave Example

In order to demonstrate the estimation algorithm developed in Section 4.3, an extension of the example in Section 3.5.3 is considered. The modal representation of the wave process is described by (3.212) through (3.216). At distinct spatial points distributed in the spatial domain, observations are taken by point sensors imbedded at each point. They are

$$z(t,i) = u(t,i) + v(t,i)$$

$$= [\phi_1(t) \quad 0 \quad \phi_2(t) \quad 0] \begin{bmatrix} x_1(i) \\ x_2(i) \\ x_3(i) \\ x_4(i) \end{bmatrix} + v(t,i) \tag{4.135}$$

$$0 \le t \le T \qquad i = 1, 2, \ldots m$$

where $\{\phi_j(t)\}$ is given by $\{\sqrt{\frac{2}{T}}(\cos\omega_j t - \sin\omega_j t)\ ;\ j = 1, 2\}$. The objective is to obtain an estimate, $u(t,i)$, of the solution to the wave equation, based on the noisy data. It is assumed that the measurement noise is characterized by

$$E\{v(t,i)\} = 0 \tag{4.136}$$

$$E\{v(t,i)v(t',j)\} = r(i)\delta(t-t')\delta_{ij} \tag{4.137}$$

When the measurement process is assumed to be available at equally spaced points the filtering algorithm described by (4.69) through (4.85) can be applied together with the discrete-continuous algorithm presented in Section 2.4. The corresponding matrices and vectors are defined as follows:

$$A = \mathrm{diag}\left[\begin{array}{cc|cc} 0 & \omega_1/c & 0 & \omega_2/c \\ -\omega_1/c & 0 & -\omega_2/c & 0 \end{array}\right] \qquad D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \tag{4.138}$$

$$Q(s) = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} \qquad R(i) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \tag{4.139}$$

$$\Phi(i) = \mathrm{diag}\left[\begin{array}{cc|cc} \cos\frac{\omega_1 d}{c} & \sin\frac{\omega_1 d}{c} & \cos\frac{\omega_2 d}{c} & \sin\frac{\omega_2 d}{c} \\ -\sin\frac{\omega_1 d}{c} & \cos\frac{\omega_1 d}{c} & -\sin\frac{\omega_2 d}{c} & \cos\frac{\omega_2 d}{c} \end{array}\right] \tag{4.140}$$

$$Q(i) = \mathrm{diag}\left[\begin{array}{cc|cc} \frac{c}{4\omega_1}\left(\frac{d}{2} - \sin\frac{2\omega_1 d}{c}\right) & \frac{c}{2\omega_1}\sin^2\frac{\omega_1 d}{c} & \frac{c}{4\omega_2}\left(\frac{d}{2} - \sin\frac{2\omega_2 d}{c}\right) & \frac{c}{2\omega_2}\sin^2\frac{\omega_2 d}{c} \\ \frac{c}{2\omega_1}\sin^2\frac{\omega_1 d}{c} & \frac{c}{4\omega_1}\left(\frac{d}{2} + \sin\frac{2\omega_1 d}{c}\right) & \frac{c}{2\omega_2}\sin^2\frac{\omega_2 d}{c} & \frac{c}{4\omega_2}\left(\frac{d}{2} + \sin\frac{2\omega_2 d}{c}\right) \end{array}\right] \tag{4.141}$$

$$z(T,i) = \left[\int_0^T z(t,i)\phi_1(t)\,dt \quad \int_0^T z(t,i)\phi_2(t)\,dt\right]^T \tag{4.142}$$

$$\hat{x}(0) = 0 \qquad P(0) = p' \operatorname{diag}\left[\begin{array}{cc|cc} 1/\omega_1^2 & 1/\omega_1^2 & 1/\omega_2^2 & 1/\omega_2^2 \\ 1/\omega_1^2 & 1/\omega_1^2 & 1/\omega_2^2 & 1/\omega_2^2 \end{array}\right] \tag{4.143}$$

The filtered estimate of u(t,s) is

$$\hat{u}(t,s) = \hat{x}_1(s)\sqrt{\tfrac{2}{T}}\,(\cos\omega_1 t + \sin\omega_1 t) + \hat{x}_3(s)\sqrt{\tfrac{2}{T}}\,(\cos\omega_2 t + \sin\omega_2 t) \tag{4.144}$$

and the mean squared error is

$$E\left\{[u(t,s) - \hat{u}(t,s)]^2\right\} = p_{11}(s)\tfrac{2}{T}\,(\cos\omega_1 t + \sin\omega_1 t)^2 + p_{33}(s)\tfrac{2}{T}\,(\cos\omega_2 t + \sin\omega_2 t)^2 \tag{4.145}$$

where $p_{11}(s)$ and $p_{22}(s)$ are the diagonal elements of P(s). The parameters used in the computer simulation were as indicated below:

$r = q = 1.0$ $\qquad$ $0 \le t \le 1.0$

$\omega_1 = 4\pi$ $\qquad$ $0 \le s \le 1.0$

$\omega_2 = 6\pi$ $\qquad$ $c = 1.0$

$m = 20$ $\quad$ $d = 0.05$

$DS = 0.01$ $\quad$ $DT = 0.01$

$$x(0) = \begin{bmatrix} 10.0 \\ -1.0 \\ 5.0 \\ 1.0 \end{bmatrix} \qquad P(0) = \begin{bmatrix} 1.0 & 1.0 & 0.0 & 0.0 \\ 1.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.25 \\ 0.0 & 0.0 & 0.25 & 0.25 \end{bmatrix}$$

where DS denotes the spatial integration interval.

The results of the simultaion are shown in Figure 3 and Figure 4 where the actual and estimated values of $u(t,s)$ are indicated with the corresponding mean squared error variance at $t=2/3$. When the accuracy of the estimate is particularly of concern, the fixed-interval smoothing algorithm demonstrated in the previous example may be used. The discontinuities at each spatial point where the temporal observation process is available can be smoothed by this operation.

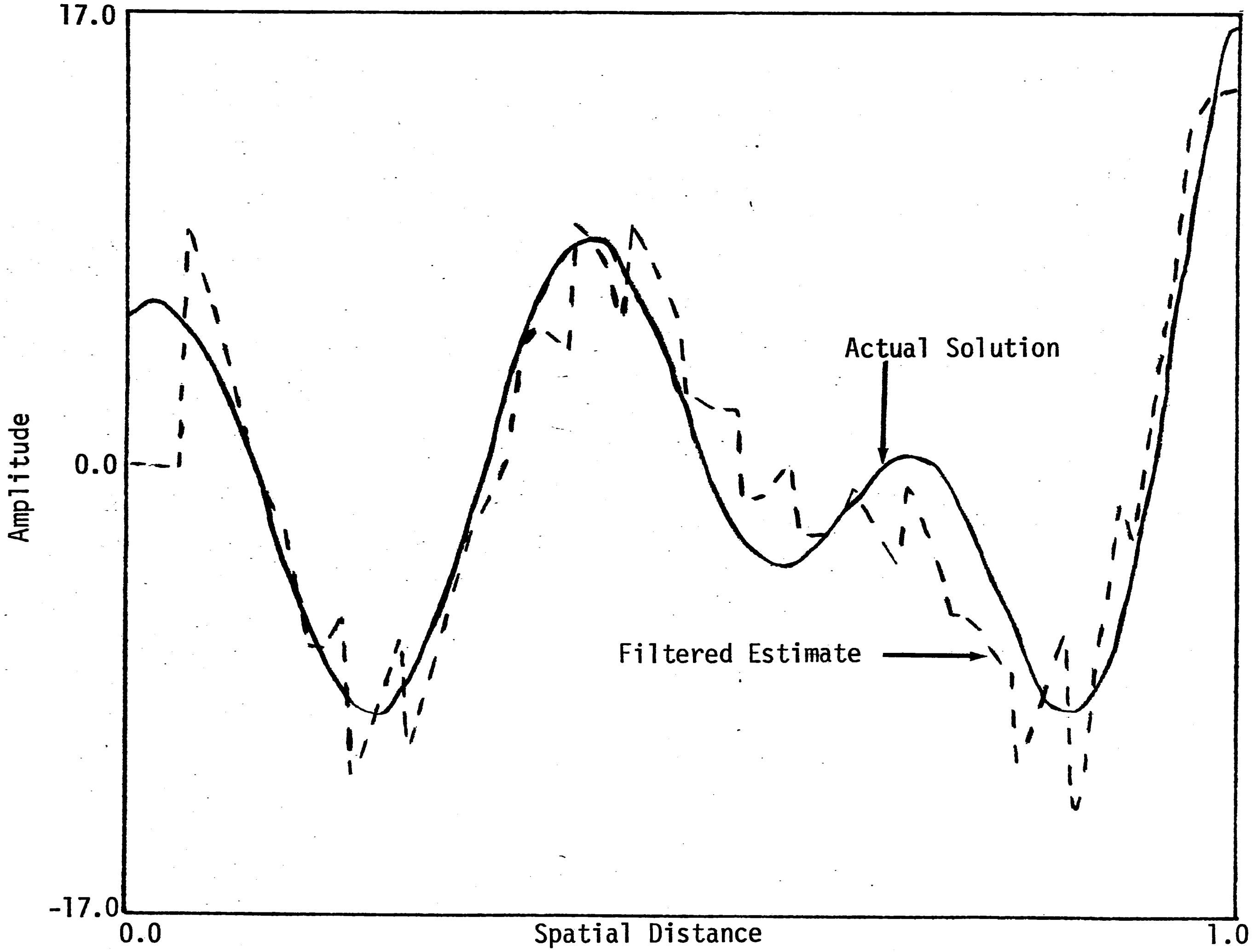


Figure 3. Actual and Estimated Value of u(t,s)
for Wave Problem

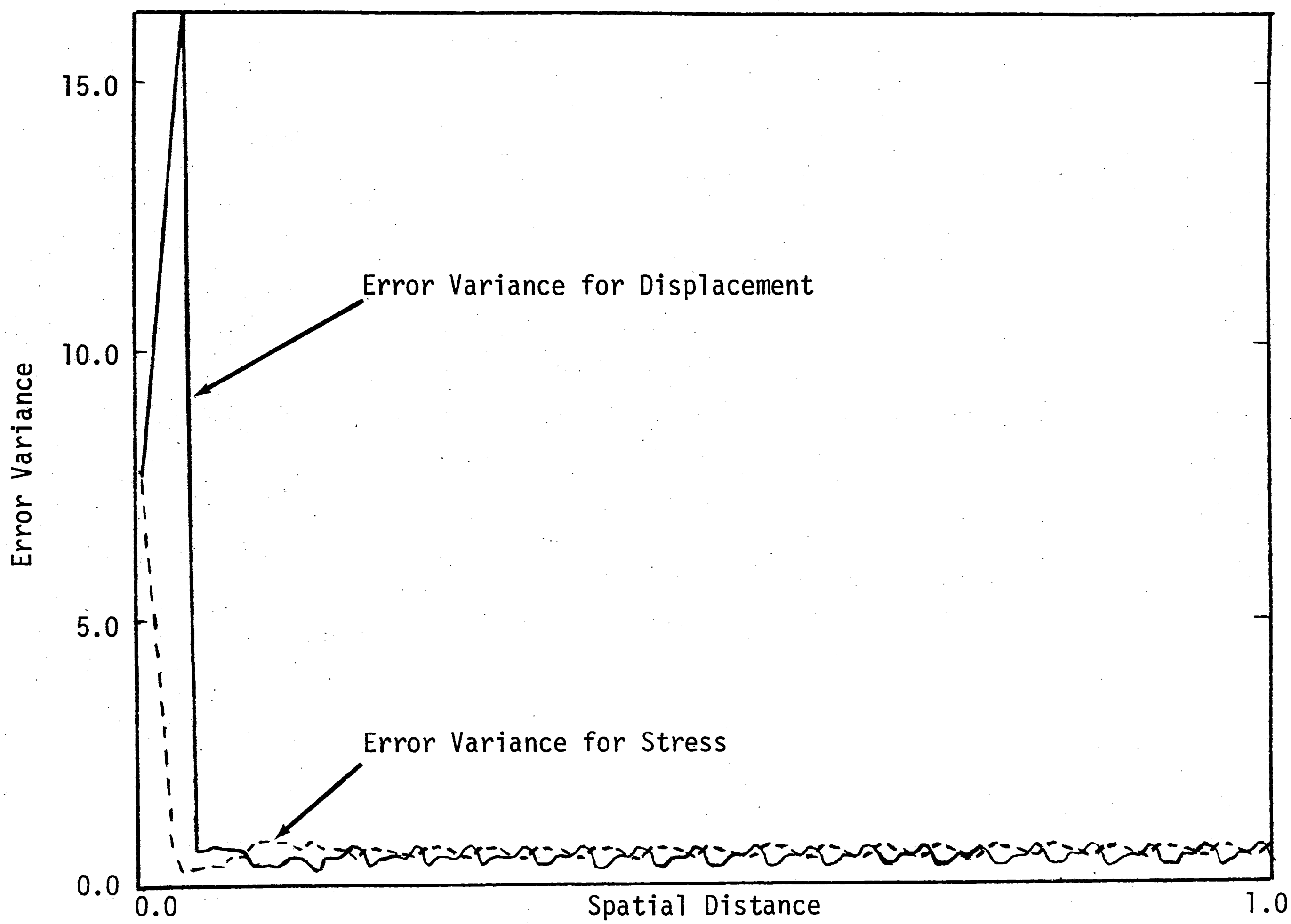


Figure 4. Mean Square Error for Wave Problem

4.5 Summary

In this chapter two distinct, but interrelated, classes of estimation problems are studied. The first is the extension of the linear state estimation techniques for lumped parameter systems to the temporal mode representation of distributed parameter systems with a set of discrete-space and continuous-time observation. The second is the estimation problem for a spatial markov-process model with the same type of observation sets. An analytical treatment of the problem with the mean squared error criterion for the first problem reveals that the Kalman filtering algorithm is the best choice. Two sets of algorithms, one for stationary observation processes and the other for non-stationary observation processes, have been presented for the second class of problems.

It was shown that a bound on mean squared error can be derived, which provides useful information in selecting the number of modes to be considered.

Two illustrative examples have been used to demonstrate the features of the methodology developed in this chapter. The results developed in this chapter makes use of state estimation theory and the series representation of stochastic distributed systems. It appears that this should have applications in many circumstances where noisy sensors are placed at discrete locations, recording physical processes as a function of time.

CHAPTER V

APPLICATIONS TO SEISMIC DATA PROCESSING

5.1 Introduction

In this chapter, the ideas of modal estimation are applied to problems in seismic signal processing. The problem of seismic prospecting is the interpretation of the seismic data on which conclusions regarding the structure of the geological medium are based. Although the physical laws involved in the understanding and analysis of seismic wave phenomena has been studied extensively for many years [35-40], the mathematical modeling and analysis of the message process for the processing of seismograms has had little attention. The inherent complexities encountered in dealing with inhomogeneous geological structures and associated random phenomena have motivated statistical approaches to the system in order to be able to describe the complicated physical process with a certain degree of generality. The deconvolution of seismic processes, investigated in [41,42,43], for the purpose of recovering the original signal having undergone convolution through a seismic section of interest is a common analysis method. In this study the underlying physics of wave propogation becomes the basis of modeling and analysis. In Section 5.2, a stochastic wave equation with random disturbances and with random one-point boundary conditions is formulated to describe wave motion when an elastic medium is disturbed by an explosive signal source or by a vibratory signal source.

There are a number of possible geophysical engineering objectives that could pertain to the analysis of seismograms. Probably the most frequent objective relates to damage criteria, determination of vibration maxima, the studies of actual medium properties and of wave propogation, and determination of the time of first arrival of shear energy. In these cases, there is a need to restore the useful signal components from the recorded seismogram corrupted by the noise present in the message process and in the measurement process. The state estimation algorithms studied in Chapter IV may be employed to accomplish this task. After some explanation is made in Section 5.3 the simulation results of the experimentation performed with a synthetic reflected seismogram is demonstrated in Section 5.4. The experiment is designed to test the hypothesis regarding whether the recording of the reflected wave contains the signal component or not. The hypothesized arrival time of the signal may be determined in an iterative way by analyzing the estimate at various spatial location.

5.2 The Wave Equation for an Elastic Body

Hooke's Law is employed to reduce the equation of motion for an elastic body to observable variables. When Newton's Second Law of Motion is applied to an elastic body, the result is:

$$\frac{\partial}{\partial s}\gamma(t,s) = (s)\frac{\partial}{\partial t^{2}}u(t,s) \qquad (5.1)$$

where γ is stress, u is displacement and $\rho(s)$ is the density. Since it is difficult to measure the stress, let alone its spatial gradient, the stress in the equation of motion can be replaced by the Hooke's Law equivalent, i.e.,

$$\gamma(t,s) = \lambda(s) \frac{\partial}{\partial s} u(t,s) \tag{5.2}$$

where $\lambda(s)$ is the modulus of elasticity.

The strain $\frac{\partial u}{\partial s}$ is itself the rate of change of the displacement of any specified portion of the elastic body, e.g., change in length per unit length. Displacements can, of course, be measured, and the resulting equation of motion in terms of displacement is given by:

$$\frac{\partial}{\partial s} [\lambda(s) \frac{\partial}{\partial s} u(t,s)] = \rho(s) \frac{\partial^2}{\partial t^2} u(t,s) \tag{5.3}$$

If the density ρ and the elastic moduli λ are positive constants, (5.3) becomes

$$\frac{\partial^2}{\partial t^2} u(t,s) = \frac{\lambda}{\rho} \frac{\partial^2}{\partial s^2} u(t,s) \tag{5.4}$$

A random forcing function w(t,s) is added to (5.4) in order to model the microseisms, the random scattering effect of the inhomogeneous medium and the near-surface multiple reflections, called reverberations, which are caused by reflections between discontinuities at near-surface and the surface of the earth. Under certain assumptions the sum of these behave as a Gaussian process [37]. It may be assumed that w(t,s) is a zero mean white Gaussian random field with variance description,

$$E\{w(t,s)\, w(t',s')\} = q\delta(t-t',\, s-s')$$

$$-\infty < t,t' < \infty\,;$$

$$0 \leq s,s' \tag{5.5}$$

Then, the inhomogeneous random wave equation becomes

$$\frac{\partial^2}{\partial s^2} u(t,s) = \frac{\rho}{\lambda}\frac{\partial^2}{\partial t^2} u(t,s) + \frac{\rho}{\lambda} w(t,s)$$

$$-\infty < t < \infty\,;\; s \geq 0 \tag{5.6}$$

where the random boundary conditions at a spatial point $s=s_1$, are assumed to be described statistically.

$$E\left\{\begin{bmatrix} u(t,s_1) \\ \frac{\partial}{\partial s} u(t,s)\Big|_{s=s_1} \end{bmatrix}\right\} = 0 \tag{5.7}$$

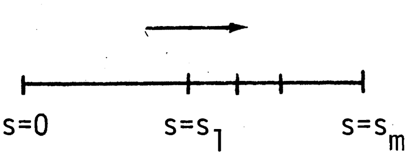
$$\mathrm{Cov}\left\{\begin{bmatrix} u(t,s_1) \\ \frac{\partial}{\partial s} u(t,s)\Big|_{s=s_1} \end{bmatrix},\ \left[u(t,s_1)\ \ \frac{\partial}{\partial s}u(t,s)\Big|_{s=s_1}\right]\right\} = \bar{P}_b(t,t') \tag{5.8}$$

The above is equivalent to the model described by (3.29) to (3.32) where c^2 is defined as λ/p. The preceding statistical properties of the unknown boundary condition may be obtained from the controlled signal source when a vibrating signal source is used at s_1 or from the direct wave signals recorded at the near-source seismic sensors when an explosive signal source is adopted. For simplicity of the treatment only a one dimensional component of the shear energy is considered. This is accomplished in the mathematical description of the particle displacement by assuming that all motion is parallel to the vertical or horizontal coordinate system.

5.3 State Estimation for Seismic Waves

The solution of the random wave system described by (5.6), (5.7) and (5.8) can be specified in a statistical manner using (3.33) and (3.34). For a particular sample, it is necessary to have some measurement of the actual realization and to process it appropriately to obtain an estimate of the behavior.

In seismic exploration measurements, the objective is to detect and record the vibratory motion of the ground or the structure caused by forces that are variable in magnitude and in direction. Since, in this study, the vector motion of the elastic body is assumed to have only one dimensional spatial component, the m equally spaced seismic sensors are distributed on a straight line from the spatial location $s=s_1$ to $s=s_m$, i.e.



When the Electromagnetic Velocity Seismograph [44] is used to record the velocity component of the reflected seismic wave, the measurement equation is of the form

$$z(t,s_i) = b_i \frac{\partial}{\partial t} u(t,s_i) + v(t,s_i) \quad i = 1,2,\ldots m \tag{5.9}$$

where b_i is a real constant of i-th sensor and $v(t,s_i)$ is assumed to be zero mean white Gaussian noise which corresponds to the seismometer noise. The modal representation of $u(t,s_i)$ gives the steady-state observational model

$$z(t,i) = b_i \sum_{j=1}^{n} \dot{\phi}_j (t)\, x_{2j-1}(s_i) + v(t,s_i) \tag{5.10}$$

or equivalently

$$z(t,i) = \sum_{j=1}^{2n} C_j(t,i)\, x_j(i) + v(t,i) \tag{5.11}$$

where

$$C_j(t,i) = b_i\, \phi_{(j+1)/2} (t),\ j = 1,3,\ldots 2n-1$$

$$C_j(t,i) = 0 \tag{5.12}$$

elsewhere, (5.11) is conveniently written by using vector notation, i.e.

$$z(t,i) = C(t,i)\ x(i) + v(t,i),\ i = 1, 2, \ldots m \tag{5.13}$$

The measurement noise, $v(t,i)$, is assumed to be zero mean white Gaussian noise such that

$$E\ \{v(t,i)\ v(t,k)\} = r(t,i)\ \delta(t-t')\delta_{ik} \tag{5.14}$$

Now, the application of the algorithms discussed in the preceding chapters to the seismic problem is of interest. The first step is to find the modal representation of the random seismic wave process, (5.6) to (5.8), under the hypothesis that the covariance at $s=s_1$, (5.8), is periodic with a period T, i.e. $\bar{P}_b(t-t') = \bar{P}_b(t-t'+T)$. This is achieved in Section 3.5.2 where a system of random ordinary differential equations is described by (3.194) through (3.202). The second step is to apply the state estimation algorithms for spatial mode representation, studied in Section 4.3. As discussed before for a general D.P.S., the observation process for seismic exploration is also constrained to be discrete-space and continuous-time. Accordingly, the discretization of the spatial model and the development of corresponding estimation schemes with a temporal preprocessor are necessary procedures to follow. In Section 4.3.1, the temporal preprocessor for stationary measurement noise is shown to be a correlator which provides a simple memoryless processor given an observation set of one time period T. The Kalman filter for estimating constants has been presented in Section 4.3.2 as the optimum temporal preprocessor with memory for a given measurement process at any instant $t \geq 0$.

At each spatial location $s_1, s_2, \text{---} s_m$, the discrete estimation algorithms described by (4.75) through (4.85) or by (4.96) through (4.109) can be used to obtain the estimate of the solution. If the estimate of the state at locations between observations is of interest, the continuous-discrete algorithm presented in Section 2.4.1 may be applied for interpolation. The fixed interval smoothing algorithm can provide a better estimate as well as an extrapolation of the state estimate at spatial locations between $s = 0$ and $s = s_1$ where the observation process is not available. This can be accomplished by assuming that the a priori statistics of the state at $s = 0$ are known.

The application of specific algorithms to various problems should depend on the relative cost of each computational algorithm. For example, if the determination of the first arrival time of shear energy is to be investigated, the Kalman filter for estimating a constant at discrete spatial locations, is sufficient to be used as an estimation tool. The problem is illustrated and simulated in the following section.

5.4 Example

In the solution of a variety of problems in seismic exploration, there is a need to determine the time of first arrival of shear energy. For a simple problem to show an application of the algorithms discussed in the previous sections, it is assumed that the arrival time of the seismic shear energy is hypothesized in interpreting a seismogram recorded at a vertical depth $s = s_1 = 0$. It is assumed that far within the earth, below $s = 0$, there is an extended boundary parallel to the earth's surface. The Figure 3 shows a simple case of a vertical seismic situation.

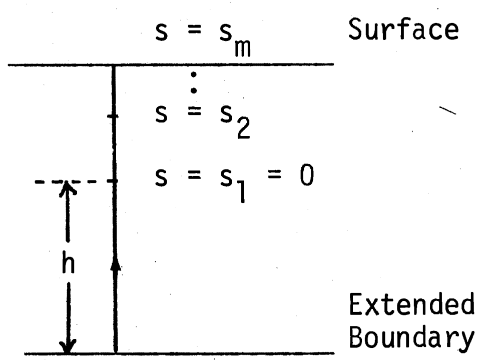


Figure 3. A Vertical Seismic Situation

If m seismic traces recorded at each spatial locations, s_1, s_2, ... s_m, are available, using the proposed signal processing algorithm, one may obtain useful information for interpreting the seismic wave behavior. When the time of first arrival of reflected shear energy is the point of interest, the time delay of the reflected wave front appearing in the sismogram of each sensor needs to be taken into consideration. The solution of the wave equation is zero at location s_i until $t \geq \frac{s_i}{c}$ where c is the wave velocity. Hence the observation at location s_i is:

$$z(t,i) = \frac{\partial}{\partial t} u_0\left(t - \frac{s_i}{c}\right) + v(t,i) \qquad t \geq \frac{s_i}{c} \tag{5.15}$$

$$= v(t,i) \qquad \text{otherwise} \tag{5.16}$$

The signal $u_0(t)$ is assumed to be a random time harmonic process at the spatial location $s = s_i = 0$ with the covariance kernel

$$P_b(t-t') = \sum_{j=1}^{2} \cos 2\pi j(t-t') \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \tag{5.17}$$

According to the algorithm derived in Section 3.5.3, the integral equation to be solved for the set of coordinate function is

$$P_j(0)\phi_j(t) = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \sum_{i=1}^{2} \int_0^1 \cos 2\pi i(t-t')\phi_j(t')\, dt' \tag{5.18}$$

Solving (5.18), one obtains only two non-zero coordinate functions,

$$\left\{\phi_1(t),\ \phi_2(t)\right\} = \left\{\cos 2\pi t + \sin 2\pi t,\ \cos 4\pi t + \sin 4\pi t\right\} \tag{5.19}$$

where

$$\left\{P_1(0),\ P_2(0)\right\} = \left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\} \tag{5.20}$$

Hence, the state $u_0(t)$ at $s = s_1 = 0$ may be expanded as

$$u_0(t) = \sum_{j=1}^{n} x_{2j-1}(0)\phi_j(t)$$

$$= \sum_{j=1}^{2} x_{2j-1}(0)\ (\cos 2\pi jt + \sin 2\pi jt) \tag{5.21}$$

which yields the measurement equation of the form

$$z(t,i) = \sum_{j=1}^{2} x_{2j-1}\left(\frac{s_i}{c}\right)\phi_j(t) + v(t,i) \tag{5.22}$$

after the arrival time of the wave, $t \geq \frac{s_i}{c}$. In the vector form

$$z(t,i) = C(t)x(i) + v(t,i) \qquad t \geq \frac{s_i}{c} \tag{5.23}$$

and

$$z(t,i) = v(t,i) \qquad t < \frac{s_i}{c} \tag{5.24}$$

where

$$C = [\dot{\phi}_1(t) \quad 0 \quad \dot{\phi}_2(t) \quad 0] \tag{5.25}$$

From (4.45) through (4.52) the message model of this example is described as

$$x_j(i+1) = \begin{bmatrix} \cos\frac{\omega_j d}{c} & \sin\frac{\omega_j d}{c} \\ -\sin\frac{\omega_j d}{c} & \cos\frac{\omega_j d}{c} \end{bmatrix} x_j(i) \qquad j = 1, 2 \tag{5.26}$$

where the one-point boundary condition is characterized by

$$E\{x_j(0)\} = 0 \tag{5.27}$$

$$E\left\{x_j(0)x_k^T(0)\right\} = P_j(0) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{5.28}$$

and $d = s_{i+1} - s_i$.

A computer simulation was carried out with the following parameters

$$\omega_1 = 2 \qquad r = 1.0$$

$$\omega_2 = 4 \qquad T = 1.0$$

$$c = 1.0 \qquad h = 0.9$$

$$d = 0.1 \qquad x(-h) = [5.0 \;\; -1.0 \;\; 3.0 \;\; 2.0]^T$$

Figures 6 and 7 show the filtered estimate of displacement at a buried sensor location and a surface location with the corresponding mean square error shown in Figure 8. Only two spatial measurements were taken. The plot of the estimate at $s = s_1$ and $s = s_2$ (at ground surface) shows that the filtered estimates converge to the actual values quickly. The filtered estimate of the seismogram at the ground surface has little filtering activity until the sensor starts picking up the travelling wave at the hypothesized arrival time, $t = 0.1$. This implies that the hypothesized arrival time of the reflected travelling wave is the true value or lies within the limit of an

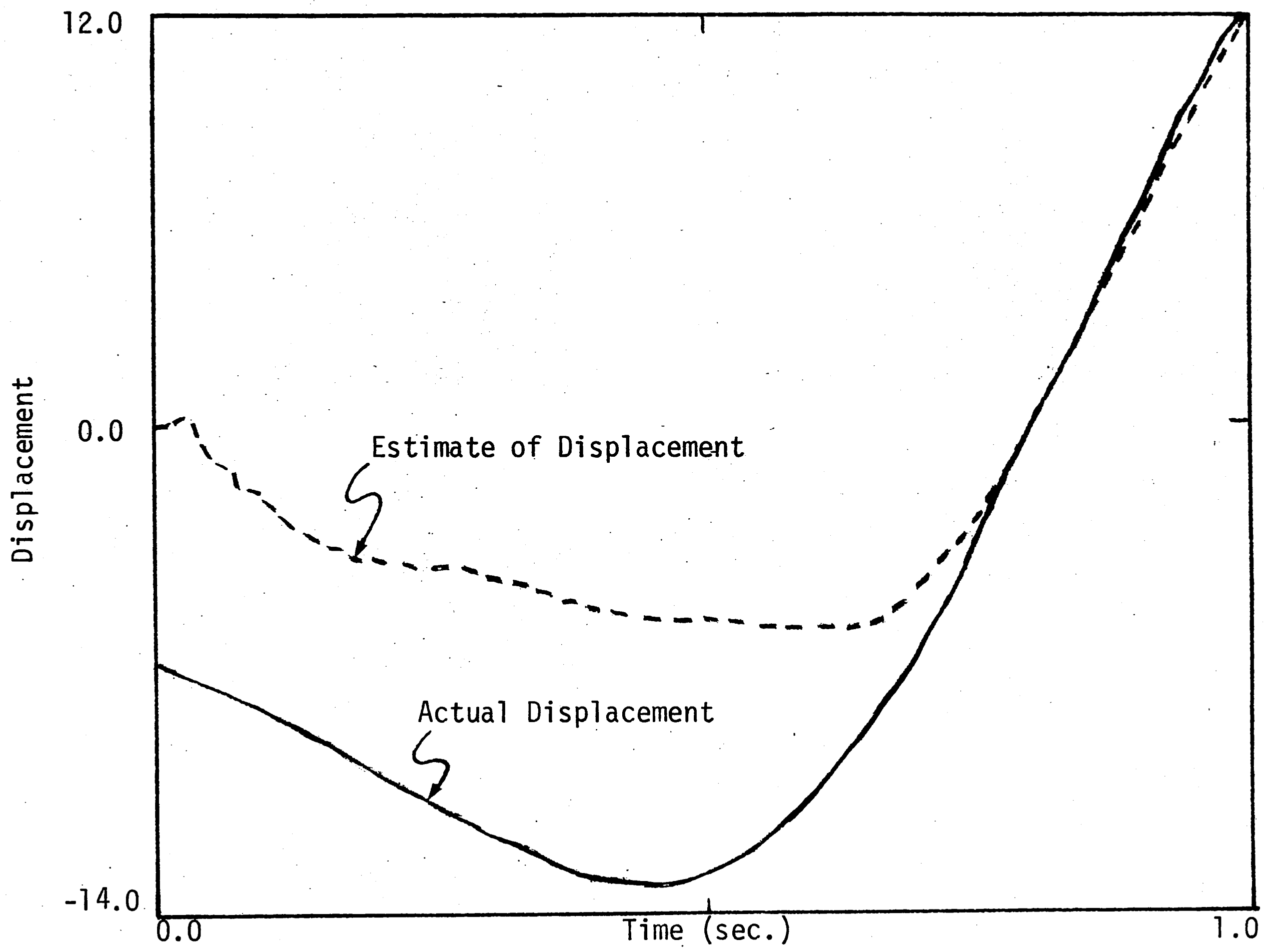


Figure 6. Filtered Estimate at the Buried Sensor

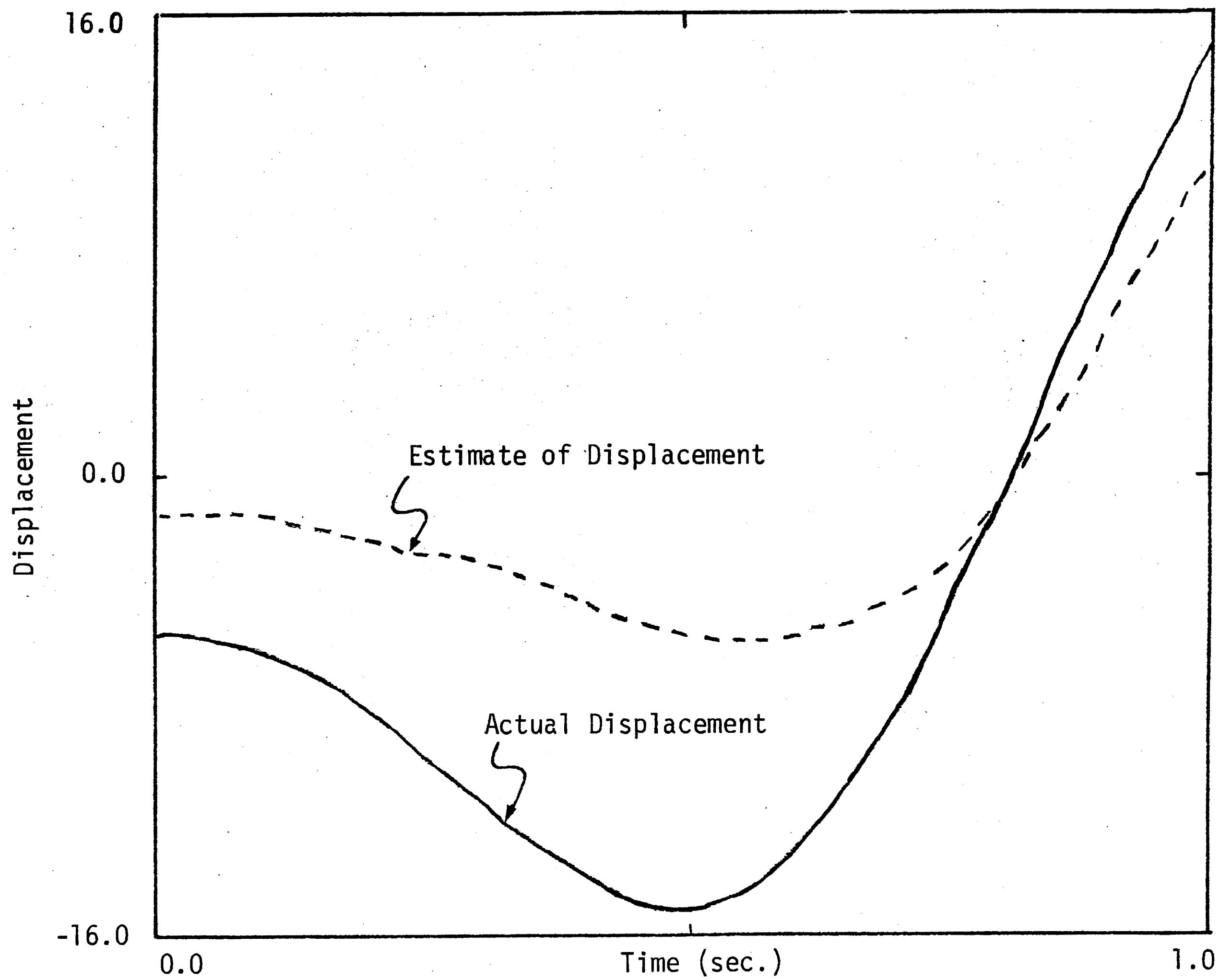


Figure 7. Filtered Estimate at the Surface Sensor

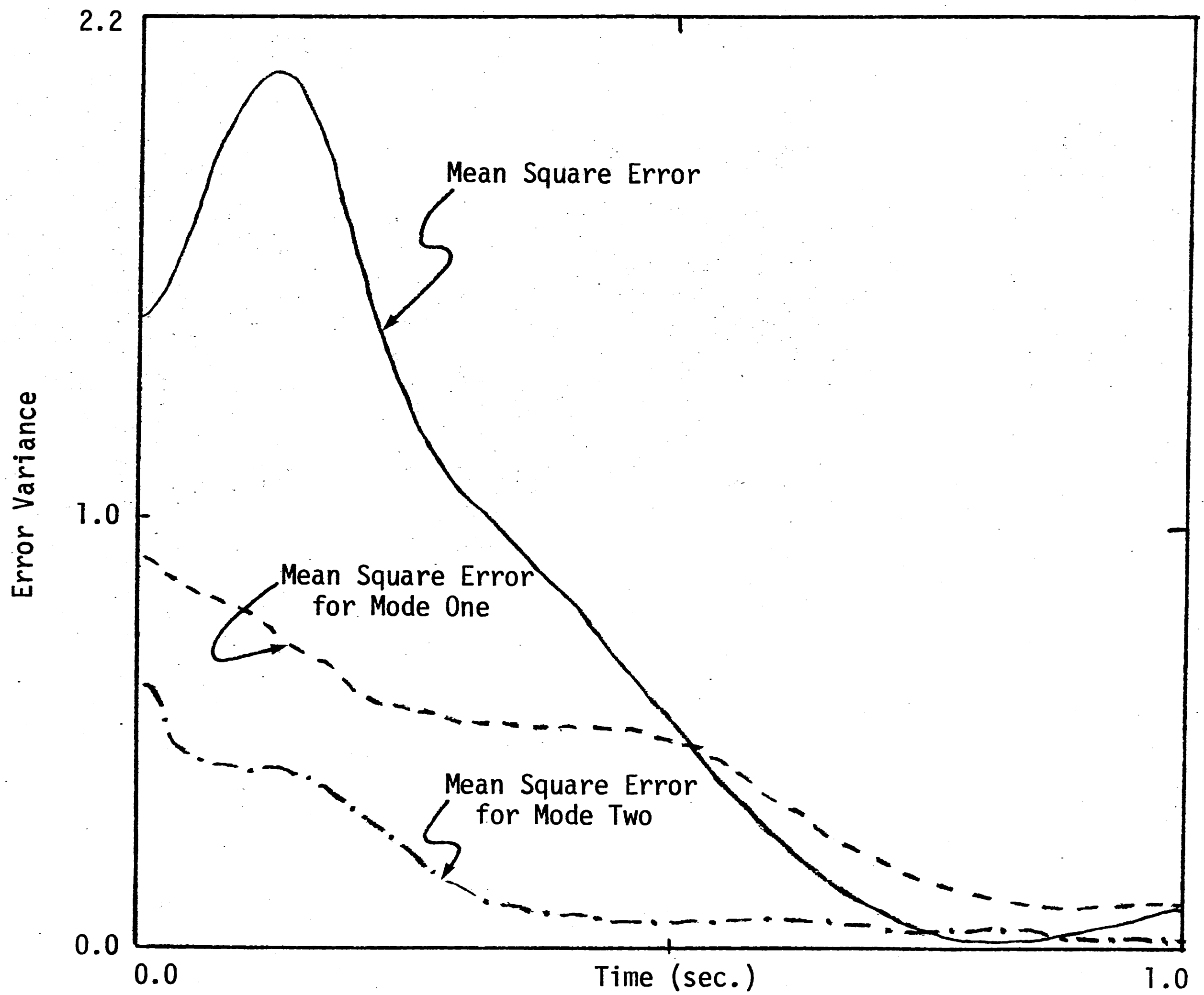


Figure 8. Mean Square Error of Estimation

acceptable error range. If an incorrect arrival time was chosen, the resultant time delay of the filtered estimate at the surface would express a certain amount of deviation from the predicted time delay. This will indicate that one should rehypothesize the arrival time of the shear energy. Of course a greater number of equally spaced buried sensors would result in better accuracy of the hypothesis test and closer prediction of the new hypothesis.

The algorithm can be applied to more realistic problems where the seismogram of interest is taken from the sensors distributed on a horizontal surface. In this case, the seismic signal to be processed will be the horizontal component of the velocity reflected from the extended boundary layer located underground, parallel to a ground surface. A nonlinear transformation of the independent parameter in addition to the state estimation technique developed in this study can be used to solve the problem, (in an approximate way due to the nonlinearity).

5.5 Summary

This Chapter proposes a mathematical model for seismic wave processes where a random time harmonic process was assumed at a certain point in space. From results obtained in previous Chapters, it was shown that state estimation algorithms may be applied to a random seismic wave process described by a stochastic inhomogeneous wave equation. Although the numerical example may not be very realistic, the significance of the study lies in the fact that a stochastic harmonic analysis has been used for the purpose of developing signal processing schemes for a random seismic situation. It is believed that this concept should prove to be useful in the area of seismic exploration.

CHAPTER VI

CONCLUSIONS

6.1 Summary of Results

The objective of the investigation was to develop state estimation algorithms for a class of distributed parameter systems including random diffusion or heat processes and wave processes. One dimensional initial-boundary-value problems with random initial conditions and white Gaussian random disturbances were considered for heat or wave processes. The observation model was assumed to be continuous in time and discrete in space.

In finding the optimum filtering algorithms, in the sense of minimum mean square error, given the prescribed model, there is a twofold problem involving both an approximation and an estimation. Fortunately, both areas are within the framework of Hilbert space which has the orthogonal projection property and consequently is ideal for the use in approximation theory as well as in estimation theory. The main feature of this study was to investigate both problems simultaneously by developing the optimum set of basis functions to be used in approximating the state of the random D. P. S. and by finding the appropriate estimation schemes to minimize the mean square error criterion.

The preliminary material has been presented in Chapter II. Green's function solution form and D'Alembert's solution form were introduced to be used in finding moment equations for random distributed systems.

In Chapter III, stochastic distributed parameter systems involving second-order random partial differential equations have been suggested for diffusion or heat processes and wave processes. General forms of randomness have been included by modeling random initial conditions, stochastic noise at the boundaries, and random disturbances as a forcing term. Since the only analysis accessible before processing the measurement processes is the characterization of the a priori statistics of the message model, the first and the second moment equations were derived. To motivate the modal representation for random distributed systems, the series representation of random fields was studied. It was shown that a white random field may be represented in terms of any orthonormal set of basis functions.

As the first step of the estimation procedure, the modal representation of the random initial-boundary-value problem and of the one-point boundary-value problem were achieved by projecting the integral form of the solution onto the subspace spanned by the optimum set of basis function. The optimum set of coordinate functions was to be the set of orthonormal eigenfunctions of the homogeneous integral equation with the kernel of the initial covariance function or of the one-point boundary covariance function. It is important to note that the characterization of the optimal set to be used in decomposing a random D. P. system into a set of random lumped parameter systems reveals a unique attribute of the study resulting from the simultaneous consideration of approximation and estimation in a single performance criterion. This formulation also provides bounds on errors arising from orthogonal projection for approximation and estimation, which is an attractive quality for many practical purposes. The bound of energy or power in uncertainty which is generated in the overall estimation procedure may be obtained prior to the actual processing of data to determine how many modes should be considered.

The same analysis was repeated for the random one-point boundary-value problem related to a time harmonic wave process, yielding a spatial mode representation for the given system.

Chapter IV consists of the development of estimation algorithms for temporal or spatial mode representations. The ordinary Kalman filtering algorithm is the optimum estimation scheme for the temporal mode message model. It is apparent that the overall mean square error constitutes the variances of neglected modes and the filtering error variance of the retained modes. Two different set of algorithms were developed for the spatial mode representation. The ordered sequential estimation algorithm was a natural outcome of the modal representation.

The basic hypothesis that the measurement process has only a finite number of modes is not relevant to this algorithm. When this observation model holds, the second algorithm making use of the Kalman filter for estimating a constant temporal process is applicable. It is valid for non-stationary obsevation noise.

Although these are not real time filtering algorithms, a conditional mean estimate is generated when the statistical characterization is assumed to be Gaussian. Two illustrative examples demonstrate the general features of a temporal model for the heat process and of a spatial model for the wave process.

A more specific application to seismic prospecting was presented in Chapter V. It was shown that the modal representation and the associated state estimation techniques developed in the previous chapters could be successfully used to achieve a state variable form for the seismic process such that the signal processing technique developed herein was applicable. Although only one specific application area was considered in this work, it is believed that there are many potential applications in the area of chemical or physical processes.

This is the first work in the field of stochastic estimation for linear distributed systems where the properties of Hilbert space are fully exploited by considering state estimation and deterministic approximation in the sense of minimum mean square error.

6.2 Suggestions for Further Research

There are many possible extensions and generalizations of this work. The dimensionality in the spatial domain may be increased to include dynamical processes evolving on a plane or in a volume. This may be achieved by considering the spatial independent parameter s as a vector quantity. In addition to increasing the spatial dimensionality of the Cartesian coordinate, spherical polar coordinates may be considered.

A natural extension of the work involves the substitution of a general self-adjoint differential operator in place of the laplacian differential operator. The resultant system would then be well-posed for the initial or boundary statistics, such as band limited spectra [23]. Various sets of orthonormal polynomials would serve as the set of basis functions for such systems. Investigation of this problem will be helpful in applying the modal estimation theory to many problems in mathematical physics.

A number of results developed for lumped parameter systems could be extended in a straightforward way to distributed systems using methods similar to those presented here. Smoothing, prediction, and filtering for colored measurement noise would be a few of these extensions. In view of growing attention, in the area of parameter identification for D. P. S. [3], it is believed that the stochastic modal analysis may have application in random parameter identification problems.

The observability and the sensor location problems studied within a deterministic framework [30] may be reformulated in a rigorous form considering stochastic observability.

To facilitate the implementation of the algorithms for multiple mode problem, it would be useful to develop hybrid computation techniques for the spatial mode filtering algorithms.

It is clear that there are many situations where partial differential equations are appropriate to describe a physical process of interest Since the equations may have uncertainties involving boundary conditions or other disturbances, it is therefore desirable to describe the situation in terms of statistics. If sensors have been placed at several spatial locations, then it may be possible that the solution to the equation can be estimated by processing the output of these sensors according to the algorithms described herein. Because of the wide range of possible applications, it is believed that the combination of approximation and estimation theory contained in this work has considerable potential value.

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